METHODS

methods

CONTENT CHECKLIST (FROM TEXTBOOKS)

○ differentiation
○ applications of differentiation
○ antidifferentiation
○ area under a curve
○ the fundamental theorem of calculus
○ the exponential function
○ calculus of trigonometric functions
○ discrete random variables
○ bernoulli & binomial distributions

$\}$ UNIT 3

○ logarithmic functions
○ calculus involving logarithmic functions
○ continuous random variables
○ the normal distribution
○ random sampling
○ sample proportions

$\}$ UNIT 4

CHAIN RULE

$y = (2x - 1)^5$

$\therefore y = u^5$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$\frac{dy}{du} = 5u^4$

$\therefore \frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{dy}}{dx}$ these cancel out

chain rule format

$\therefore \frac{dy}{dx} = 5u^4 \cdot 2$

$\frac{du}{dx} = 10u^4$

trying to find dy/dx not du/dx

$10(2x - 1)^4$

POWER RULE FOR CHAIN RULE

$y = (2x - 1)^5$

$y' = 5(2x - 1)^4 \cdot 2$

in general

if $y = [f(x)]^n$

derivative of function inside bracket ↓

then $\frac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$

$y = \sqrt[3]{4 - 3x^3}$

$= (4 - 3x^3)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}(4 - 3x^3)^{-2/3} \cdot (-9x^2)$

can simplify

$= -3x^2(4 - 3x^3)^{-\frac{2}{3}}$

$= \frac{-3x^2}{(4 - 3x^3)^{2/3}}$ ← simplified to positive indices

$= \frac{-3x^2}{\sqrt[3]{(4 - 3x^3)^2}}$

$y = (2x^2 - 3x)^{1/2}$

$y' = \frac{1}{2}(2x^2 - 3x)^{-1/2} \cdot (4x - 3)$

$= \frac{(4x - 3)}{2(2x^2 - 3x)^{1/2}}$

$= \frac{4x - 3}{2\sqrt{2x^2 - 3x}}$

PRODUCT RULE

$y = f(x) \cdot g(x)$

$$\frac{dy}{dx} = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

EXAMPLE

$$y = \overset{f}{3x^2}\ \overset{g}{\left[(2x+1)^3\right]}$$

chain rule used ↓

$$y' = 6x(2x+1)^3 + 3x^2 \cdot 3(2x+1)^2(2)$$

$$= \underbrace{6x}_{}\underbrace{(2x+1)^3}_{} + \underbrace{18x^2}_{}\underbrace{(2x+1)^2}_{}$$

$$= 6x(2x+1)^2[2x+1+3x]$$ ← factored

$$y' = 6x(2x+1)^2(5x+1)$$

EXAMPLE 2

$$y = (\sqrt{x+1})(3x^2-1)$$

$$= \underset{f}{(x+1)^{\frac{1}{2}}}\ \underset{g}{(3x^2-1)}$$

chain rule ↓

$$y' = \tfrac{1}{2}(x+1)^{-1/2}(3x^2-1) + \sqrt{x+1}\,(6x)$$

$$\frac{3x^2-1}{2(x+1)^{1/2}} + \frac{6x(x+1)^{1/2} \times 2(x+1)^{1/2}}{1 \times 2(x+1)^{1/2}}$$

LCD = $2(x+1)^{1/2}$

$12x^2 + 3x^2 = 15x^2$

$$\frac{3x^2 - 1 + 12x(x+1)}{2(x+1)^{1/2}}$$

$$= \frac{15x^2+12x-1}{2\sqrt{x+1}}$$

QUOTIENT RULE

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

EXAMPLE

$$y = \frac{(x^2+1)}{(1-x)} \begin{matrix} f \\ g \end{matrix}$$

$$y' = \frac{(1-x)2x - (x^2+1)(-1)}{(1-x)^2}$$

simplify:

$$y' = \frac{2x - 2x^2 + x^2 + 1}{(1-x)^2}$$

$$= \frac{1 + 2x - x^2}{(1-x)^2}$$

same answer

or (other method not quotient rule)

$$\frac{x^2+1}{1-x}$$

$$= \overset{f}{(x^2+1)}\overset{g}{(1-x)^{-1}}$$

use product rule instead

$$y' = 2x(1-x)^{-1} + (x^2+1)[\underbrace{-1(1-x)^{-2}(-1)}_{=+}]$$

simplify:

$$(1-x) \times \frac{2x}{(1-x)}{}^{(1\times x)} + \frac{x^2+1}{(1-x)^2} \qquad LCD = (1-x)^2$$

$$\frac{2x(1-x)}{(1-x)^2} + \frac{x^2+1}{(1-x)^2}$$

$$= \frac{2x - 2x^2 + x^2 + 1}{(1-x)^2}$$

$$= \frac{1 + 2x - x^2}{(1-x)^2}$$

SIGN TEST

note:
may be asked for t·p or p·o·i

to test for a turning point ·

$y'' < 0 \rightarrow$ max
$y'' > 0 \rightarrow$ min
at $x = x_0$

POI (POINTS OF INFLECTION)

when $y'' = 0$
i·e either oblique or horizontal
↰ most common

EXAMPLE

$y = ax^4 + bx^3 - x^2 + 1$
has an inflection point @ $(1, -4)$
determine $a + b$

$y' = 4ax^3 + 3bx^2 - 2x$
$y'' = 12ax^2 + 6bx - 2$
$y'' = 0$ @ $x = 1$
i·e $0 = 12a + 6b - 2$
$y = -4$ when $x = 1$
i·e $-4 = a + b \cancel{+1} \cancel{-1}$
$\therefore a + b = -4$

substitution or elimination

$6a + 3b = 1$

elimination
$-3a - 3b = 12$
$3a = 13$
$a = 13/3$

substitution
$a = -4 - b$
Sub into $(6a + 3b = 1)$
$6(-4 - b) + 3b = 1$
$-24 - 6b + 3b = 1$
$-25 = 3b$
$b = -25/3$

$\therefore 13/3 + b = -4$
$b = -\frac{12}{3} - 13/3$
$= -\frac{25}{3}$

SUMMARY

$\frac{dy}{dx} = 0$ @ t·p's

$\frac{d^2y}{dx^2} = 0$ @ POI (points of inflection)

- a curve is always concave down if $\frac{d^2y}{dx^2} < 0$

<u>also</u> a curve is always concave up if $\frac{d^2y}{dx^2} > 0$

- if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, then a possible horizontal POI may exist

  ↳ could be t·p

  ↳ sign test to confirm

RATES OF CHANGE

$\frac{dy}{dx}$ = the rate of change of y with respect to x

but if $P = \frac{3Q+1}{2Q^2-3}$

$\frac{dP}{dQ}$ = the rate of change of P with respect to Q

ACCELERATION

rectilinear motion

- movement in a straight line
  - $x = f(t)$ - displacement
  - $v = f'(t)$ - velocity
  - $a = f''(t)$ - acceleration

or

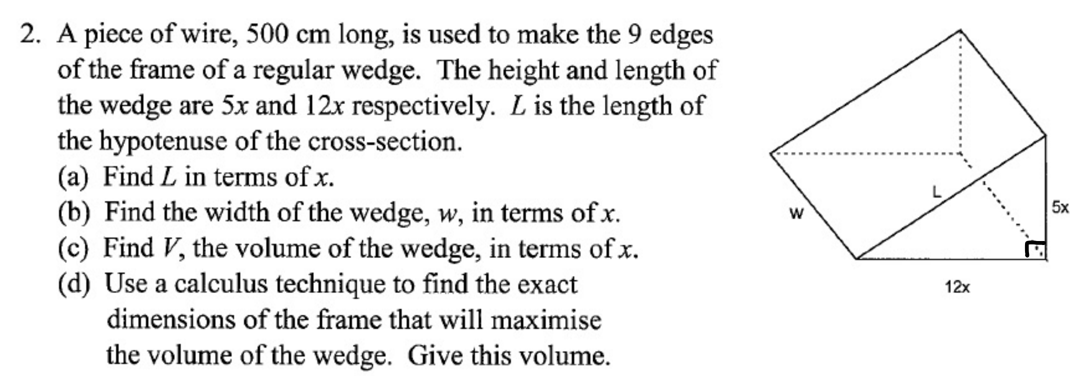
$a = \frac{d}{dt}(v(t))$

a.k.a $\frac{dv}{dt}$

$v = \dot{x}$ (derivative w/ respect to time)

$a = \ddot{x}$ (like saying $f''(t)$)

OPTIMISATION



2. A piece of wire, 500 cm long, is used to make the 9 edges of the frame of a regular wedge. The height and length of the wedge are $5x$ and $12x$ respectively. L is the length of the hypotenuse of the cross-section.
   (a) Find L in terms of x.
   (b) Find the width of the wedge, w, in terms of x.
   (c) Find V, the volume of the wedge, in terms of x.
   (d) Use a calculus technique to find the exact dimensions of the frame that will maximise the volume of the wedge. Give this volume.

a) $L = \sqrt{(12x)^2 + (5x)^2}$
$= 13x$

b) all edges = 500 cm
$500 = 3w + 2(12x) + 2(5x) + 2L$ ← $13x$
$500 = 3w + 60x$
$3w = 500 - 60x$
$w = \frac{1}{3}(500 - 60x)$

c) volume = ?
$V = \overset{10}{\cancel{20}} \times \frac{1}{3}(500 - 60x)$
$= 10x^2(50 - 6x)$
$= 100x^2(50 - 6x)$
take 2 out
$= 200\ (25x^2 - 3x^3)$

d) maximise volume
$V = 200x\ (25x^2 - 3x^3)$

① differentiate
$dy/dx = 200\ (50x - 9x^2)$

② make = 0
$0 = 200(5x - 9x^2)$
$0 = (5x - 9x^2)$
i.e. $\cancel{x}(50 - 9x) = 0$
real life, $x = 0$ makes no sense
$x = \frac{50}{9}$

③ second derivative
$\frac{d^2V}{dx^2} = 200\ (50 - 18x) < 0\ \forall x$

means 'for all'

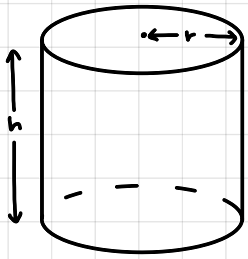
for all x positive values (negative makes no sense in this context)

④

continue working answer

example 2:

a closed cylinder is to have a volume of $1024\pi \text{ cm}^3$. find its radius if a minimum amount of sheet metal is to be used in its construction



$V = 1024\pi \text{ cm}^3$

$A = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h$

$\therefore 1024\pi = \pi r^2 h$

$1024 = r^2 h$

$h = \frac{1024}{r^2}$

plug into

$2\pi \cancel{r} \cdot \frac{1024}{r^{\cancel{2}}} + 2\pi r^2$

$= 2\pi \frac{1024}{r} + 2\pi r^2$

$A = 2\pi \left(\frac{1024}{r} + r^2 \right)$

$\frac{dA}{dr} = 2\pi \left(-\frac{1024}{r^2} + 2r \right)$

$= 0$ when $\frac{-1024}{r^2} + 2r = 0$

$-1024 + 2r^3 = 0$

$2r^3 = 1024$

$r^3 = 512$

$r = 8$ ← does this give a min? (test w/ $f''(x)$)

$\frac{d^2A}{dr^2} = 2\pi \left(\frac{2(1024)}{r^3} + 2 \right)$

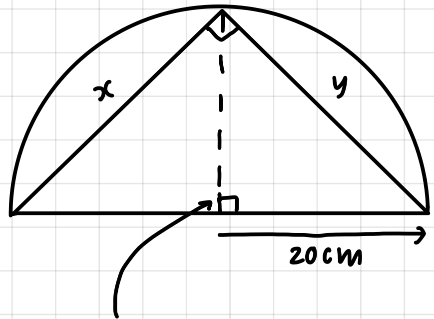
$> 0 \ \forall$ positive r

$\therefore$ minimum

$r = 8 \text{ cm}$ for min surface area

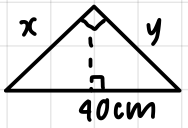
example 3:

a triangle is inscribed in a semicircle of radius 20cm so that one side is along the diameter. find the dimensions to maximise the area of the triangle



make it b so $\frac{1}{2}bh$ $\therefore \frac{1}{2}xy$

① area of triangle $= \frac{1}{2}xy$



② $40^2 = x^2 + y^2$

$y = \sqrt{40^2 - x^2}$

③ sub into each other:

$A = \frac{1}{2}x\sqrt{40^2 - x^2}$

$= \frac{1}{2}x(40^2 - x^2)^{1/2}$

④ find dA/dx w/ product rule

$f = \frac{1}{2}x$ $\qquad f' = \frac{1}{2}$

$g = (40^2 - x^2)^{1/2}$ $\qquad g' = \frac{1}{2}(40^2 - x^2)^{-1/2}$

$dA/dx = \frac{1}{2}(40^2 - x^2)^{1/2} + \frac{1}{2}x \cdot \frac{1}{2}(40^2 - x^2)^{-1/2}(-2x)$

$= \frac{1}{2}(40^2 - x^2)^{1/2} - \frac{1}{2}x^2(40^2 - x^2)^{-1/2}$

$= 0$ when $\frac{1}{2}(40^2 - x^2)^{1/2} - \frac{1}{2}x^2(40^2 - x^2)^{-1/2} = 0$

$\div \frac{1}{2}$ $\quad (40x^2 - x^2)^{1/2} - x^2(40^2 - x^2)^{-1/2} = 0$

i.e. $(40^2 - x^2)^{1/2} = \frac{x^2}{(40^2 - x^2)^{1/2}}$ $\quad \times (40^2 - x^2)^{1/2}$

$40^2 - x^2 = x^2$

$40^2 = 2x^2$

$\frac{\overset{20}{\cancel{40}} \times 40}{\cancel{2}\ 1}$

$20 \times 40 = x^2$

$x = \sqrt{20 \times 40}$

$x = \sqrt{100 \times 4 \times 2}$

$= 10 \cdot 2\sqrt{2}$

$\boxed{x = 20\sqrt{2}}$

notes from the textbook - differentiation (ch1)

THE PRODUCT RULE

$y = f(x) \times g(x)$

then

$dy/dx = g(x)f'(x) + f(x)g'(x)$

alternatively, you can use u and v to represent f(x) and g(x)

$y = uv$

then

$dy/dx = v\frac{du}{dx} + u\frac{dv}{dx}$

THE QUOTIENT RULE

$y = \frac{u}{v}$ where u and v are functions of x

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

CHAIN RULE

$y = 3x^2 + 4$, $dy/dx = 6x$

however, suppose we aren't given y directly in terms of x but instead are given y in terms of another variable e.g. u which is in terms of x

if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

example:

$y = 4u + 3$ $\quad u = x^2 - 4$

$\frac{dy}{du} = 4$ $\quad \frac{du}{dx} = 2x$

$\frac{dy}{dx} = (4)(2x)$

$= 8x$

POINTS TO NOTE

if $y = [f(x)]^n$

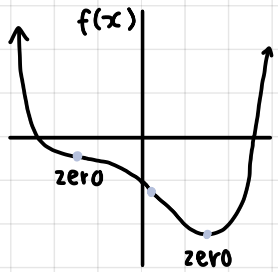
then

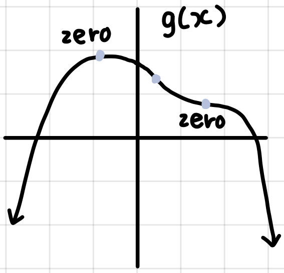
$dy/dx = n[f(x)]^{n-1} f'(x)$

notes from the textbook - applications of differentiation (ch2)

EXAMINING THE SECOND DERIVATIVE

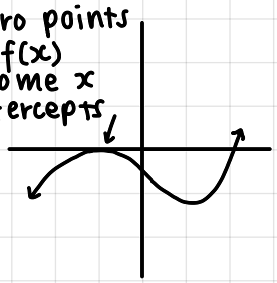
points of inflection:

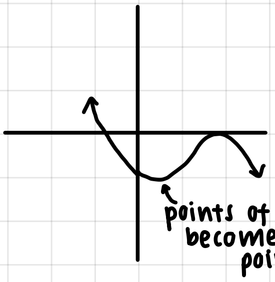




this allows a sketch of $f'(x)$ and $g'(x)$ to be drawn:

zero points on f(x) become x intercepts



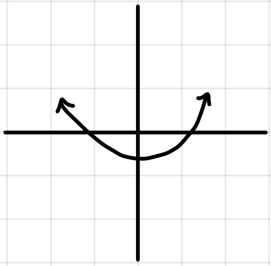


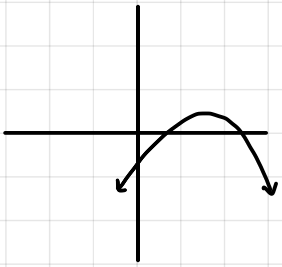
points of inflection become turning points

NOTES

- when $f''(x) < 0$ then $f(x)$
  = concave down
- when $f''(x) > 0$ then $f(x)$
  = concave up
- at all the points of inflection, $f''(x) = 0$
  (but $f''(x) = 0$ doesn't necessarily mean its a POI)

and also $f''(x)$





LOCATING T.Ps + POI

- to determine nature:
  - ↳ sign test (gradient on either side)
  - ↳ $f''(x)$
    - ↳ $y'' < 0$ → max at $x = x_0$
    - ↳ $y'' > 0$ → min
    - ↳ $y'' = 0$ → must use sign test to determine

DOT NOTATION

often $dy/dt = \dot{y}$
and $d^2y/dt^2 = \ddot{y}$

displacement
velocity
acceleration

these are all vectors & require a direction

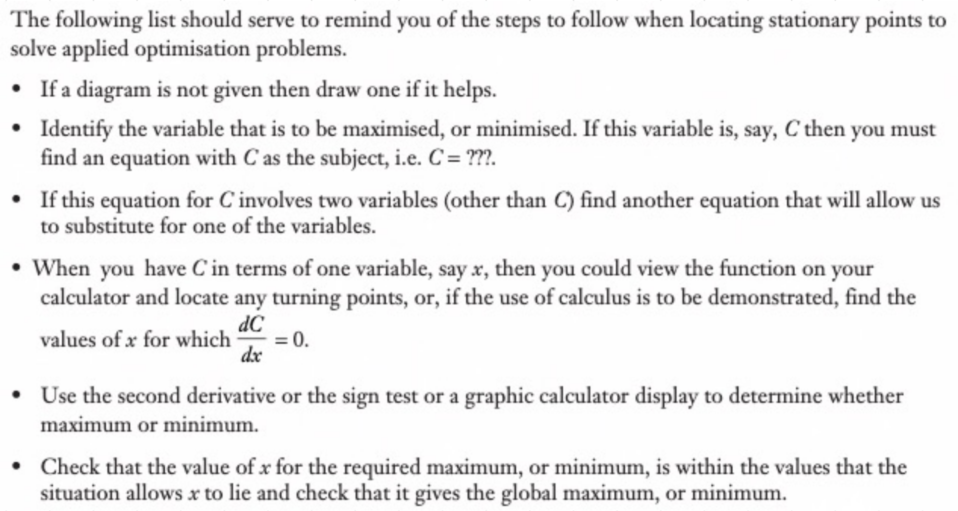
RATES OF CHANGE

V = volume
t = time
then $\frac{dV}{dt}$ = the change of volume with respect to time

ACCELERATION

if displacement is $x [= f(t)]$
the velocity as a function of time is given by: $v = \frac{dx}{dt}$

and $\frac{dv}{dt}$ = acceleration
(i.e. 2nd derivative) $\frac{d^2x}{dt^2}$



The following list should serve to remind you of the steps to follow when locating stationary points to solve applied optimisation problems.

- If a diagram is not given then draw one if it helps.
- Identify the variable that is to be maximised, or minimised. If this variable is, say, C then you must find an equation with C as the subject, i.e. C = ???.
- If this equation for C involves two variables (other than C) find another equation that will allow us to substitute for one of the variables.
- When you have C in terms of one variable, say x, then you could view the function on your calculator and locate any turning points, or, if the use of calculus is to be demonstrated, find the values of x for which $\frac{dC}{dx} = 0$.
- Use the second derivative or the sign test or a graphic calculator display to determine whether maximum or minimum.
- Check that the value of x for the required maximum, or minimum, is within the values that the situation allows x to lie and check that it gives the global maximum, or minimum.

SMALL CHANGES

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{\delta y}{\delta x}$$

lowercase delta ↓

δy ← small change in y

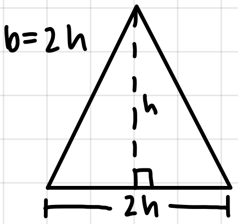
δx ← small change in x

$$\boxed{\delta y \approx \frac{dy}{dx} \cdot \delta x}$$

← increments formula/ incremental formula (on formula sheet) ← (small change formula)

EXAMPLE

if the height of a triangle changes from h = 4 cm to h = 4·1 cm & b = 2h



$$A = \frac{bh}{2} = \frac{\cancel{2}h \times h}{\cancel{2}} = h^2$$

$\therefore A = h^2$

$$\delta A = \underbrace{2h}_{\frac{dA}{dh}}\ \underbrace{\delta h}_{\text{the small change } (4\cdot1-4)}$$

$= 2(4)(0\cdot1)$

$\delta A = 0\cdot8\text{ cm}^2$ → change in area as h goes from 4 → 4·1 cm (small change in A for small change in h)

$\boxed{\doteqdot \text{ is same as } \approx}$

EXAMPLE 2

determine the approximate percentage error in the volume and SA (surface area) of a sphere of radius 4cm, corresponding to an error of 1% in its radius

volume of sphere

$V = \frac{4}{3}\pi r^3$

$r = 4\text{cm}$

$\delta V = \frac{dV}{dr} \cdot \delta r$

$\frac{\delta V}{V} \times 100 = \%$ change in V ← change in volume

$\delta V = 4\pi r^2 \cdot \delta r$

$100 \times \frac{\delta V}{V} = \frac{4\pi r^2 \cdot \delta r}{V} \times 100$ ← sub original V into this

$100 \times \frac{\delta V}{V} = \frac{\cancel{4\pi} r^{\cancel{2}} \delta r}{\cancel{4/3 \pi} r^{\cancel{3}\,1}} \times 100$

$= 3\ \frac{\delta r}{r} \times 100$ ← makes it %

↑ 1% (in Qu) i.e. $0\cdot01 \times 100 = 1$

$= 3 \times 1$

$\delta V = 3\%$

↑ % change in volume (if radius has error of 1%, volume has error of 3%)

$\frac{4\pi r^2}{1} \div \frac{4\pi r^3}{3}$

$\frac{\cancel{4\pi r^2}}{1} \times \frac{3}{\cancel{4\pi r^3}\,1}$

$= \frac{3}{r}$

surface area

$SA = 4\pi r^2$

$\delta A = \frac{dA}{dr} \cdot \delta r$

$\delta A = 8\pi r \cdot \delta r$

100 x

MARGINAL VALUE

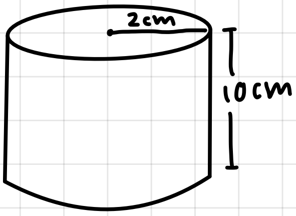
if $C(x)$ is the cost associated with producing x items then $C'(x)$ is the marginal list
i.e. the cost of producing 1 more item when x_0 have been produced

USING INCREMENTAL FORMULA

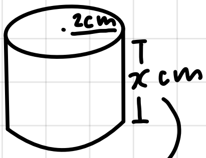
$\delta C \approx C'(x)\ \delta x$ where $\delta x = 1$
$x = x_0$

EXAMPLE (Q10 small changes worksheet)

a closed cylinder 10cm long has a cross-sectional radius of 2cm. use derivatives to find the approx. error in the volume of the cylinder corresponding to an error of
(a) 0.1cm in measurement of length



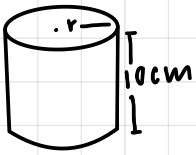
(a)



introduced variabe that has the error

$V = \pi r^2 h$
$= \pi(4)x$
$\delta V \approx 4\pi \cdot \delta x$
$= 4\pi(0.1)$
$= 0.4\pi \text{cm}^3$

(b)

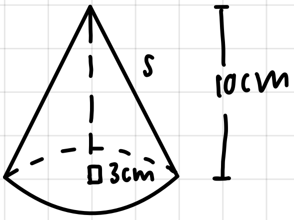


$\delta r = 0.05$

$V = \pi r^2 h$
$= \pi r^2(10)$
$V = 10\pi r^2$
$\delta V \approx 20\pi r \cdot \delta r$
$\delta V = 20\pi(2) \cdot (0.05)$
$= 2\pi \text{ cm}^3$

EXAMPLE 2 (Q9 small changes worksheet)

use derivatives to find the approximate change in the $\overbrace{\text{curved surface area}}^{\text{not circle @ bottom}}$ of a cone of height 10cm corresponding to a change in its base radius 3cm → 3.02cm



$A = \pi r s$
need formula with r's only
∴ need expression for s in terms of r

$s = \sqrt{10^2 + r^2}$
$\therefore A = \pi r\sqrt{100 + r^2}$
$\delta r = 0.02$ ↓ product rule
$\frac{dA}{dr} = \pi\sqrt{100 + r^2} + \pi r \cdot \frac{1}{2}(100 + r^2)^{-1/2}(2r)$
$\delta A \approx \frac{dA}{dr} \cdot \delta r$ $r = 3$, $\delta r = 0.02$
$= \left[\pi\sqrt{100 + r^2} + \pi r \cdot \frac{1}{2}(100 + r^2)^{2}(2r)\right](0.02)$
$= 0.71 \text{ cm}^2$

EXAMPLE 3 (Q15 small changes worksheet)

the period of oscillation of a pendulum of length L, is given by $T = 2\pi\sqrt{\frac{L}{g}}$, where g is a constant. Find the approximate percentage change in T corresponding to a 5% drop in the length of the pendulum. Find the approximate % change in L corresponding to a 2% increase in the period of oscillation

$$T = 2\pi\sqrt{\frac{L}{g}}$$
$$= \frac{2\pi}{\sqrt{g}} L^{1/2}$$

$y = 3x^2$
$y' = 2\times 3x$
$y = 3x^{1/2}$
$y' = 3\,\tfrac{1}{2}x^{-1/2}$

$$\delta T \approx \frac{2\pi}{\sqrt{g}}\cdot\frac{1}{2}L^{-1/2}\,\delta L$$
$$= \frac{\pi}{\sqrt{g}}\cdot L^{-1/2}\,\delta L$$

% change in T

$$100\times\frac{\delta T}{T} = \frac{\left(\frac{\pi L^{-1/2}}{\sqrt{g}}\right)}{\underbrace{\left(\frac{2\pi L^{1/2}}{\sqrt{g}}\right)}_{=T}}\,\delta L \times 100$$

$$= \frac{1}{2}\left(\frac{\delta L}{L}\times 100\right)$$
$$= \frac{1}{2}(-5)$$ ← 5% drop

$= -2.5\%$

∴ 2.5% decrease

$$\frac{\pi L^{-1/2}}{\sqrt{g}} \div \frac{2\pi L^{1/2}}{\sqrt{g}}$$
$$= \frac{\cancel{\pi} L^{-1/2}}{\cancel{\sqrt{g}}} \times \frac{\overset{1}{\cancel{\sqrt{g}}}}{\underset{2}{}2\cancel{\pi} L^{1/2}}$$
$$= L^{1/2}\,L^{1/2}\cdot\overset{1}{2}$$
$$= \frac{1}{2L}$$

$$\left(\frac{100\times\delta T}{T}\right) = \frac{1}{2}\left(\frac{\delta L}{L}\times 100\right)$$

↓ finding this

$2 = \frac{1}{2}(\qquad)$

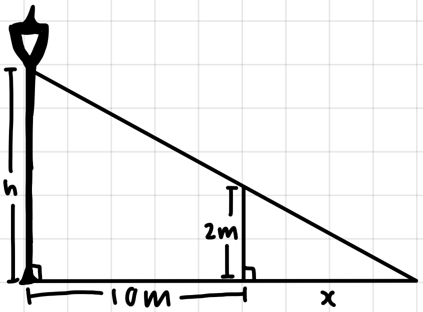
2% change in T (period)

∴ 4% increase in L

$\therefore \frac{2}{0.5} = (\quad)$

$= 4$

EXAMPLE 4 (Q12 small changes worksheet)



(changes from 2 → 2·01)

similar triangles

$$\frac{h}{2} = \frac{x+10}{x} \qquad h = \frac{2(x+10)}{x}$$

$$h = 2 + \frac{20}{x}$$

$$\delta h = \frac{-20}{x^2} \cdot \delta x$$

± 0·01 (either way)

$$= \frac{-20}{2^2}(0\cdot01)$$

$$= \overset{+}{-}0\cdot05\text{ m}$$

ANTIDIFFERENTIATION (INTEGRATION)

NO PRODUCT, CHAIN OR QUOTIENT RULE!!

EXAMPLE 1

if $\frac{dy}{dx} = \frac{x^4}{4}$ $y = ?$

experiment:
$y = x^5$
$dy/dx = 5\underline{x^4}$

$y = \frac{x^5}{20} + c$

EXAMPLE 2

if $dy/dx = \frac{1}{x^4}$
$= x^{-4}$

$y = \frac{-x^{-3}}{3} + c$

EXAMPLE 3

if $dy/dx = \sqrt{x}$ $y = ?$
$= x^{1/2}$

$y = \frac{x^{3/2}}{3/2} + c$

← same as × by 2/3

$= \frac{2}{3} x^{3/2} + c$

$\int x^n \, dx$

integrating w/ respect to x

these must match

$= \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ ← indefinite integrals

can't ÷ by 0

integrate the following

① $\int \sqrt{\frac{1}{4\sqrt{x}}} \, dx$

$\frac{1}{4} \int \frac{dx}{\sqrt{x}}$

can put constants outside of the integral

$\frac{1}{4} \int x^{-1/2} \, dx$

$= \frac{1}{\underset{2}{\cancel{4}}} \cancel{2} x^{1/2} + c$

$= \frac{1}{2} x^{1/2} + c$

② $\int \frac{x^2 + x}{x^4} \, dx$

$\int x^{-2} + x^{-3} \, dx$

$\frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + c$

$-\frac{1}{x} - \frac{1}{2x^2} + c$

③ $\int (1 + 2\sqrt{x})^2 \, dx$

$\int 1 + 4\overset{x^{1/2}}{\sqrt{x}} + 4x \, dx$

$= x + \frac{4x^{3/2}}{3/2} + \frac{4x^2}{2} + c$

$= x + 4 \frac{2}{3} x^{3/2} + \frac{4x^2}{2} + c$

$= x + \frac{8}{3} x^{3/2} + 2x^2 + c$

$$\int (3x+7)^7\,dx$$

$$= \frac{(3x+7)^8}{3\times 8} + c$$

$$= \frac{(3x+7)^8}{24} + c$$

$$\int (ax+b)^n\,dx$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

ON CLASSPAD

math2 keyboard

$$\int_{\square}^{\square} \square\, d\square$$

e.g. $\int_{\square}^{\square} (3x+7)^7\,dx$

put on standard for exact value

$$= \frac{(3x+7)^8}{24}$$

calc doesn't add a (+c) @ the end

RULE 2

$$y = [f(x)]^{n+1}$$

$$\int \frac{dy}{dx}\,dx = \int (n+1)[f(x)]^n \cdot f'(x)\ dx$$

$$y = (n+1) \int [f(x)]^n \cdot f'(x)\ dx$$

$$[f(x)]^{n+1} = (n+1) \int [f(x)]^n\ f'(x)\ dx$$

$$\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$$ — the rule

$\int \frac{dy}{dx}$ w/ respect to x
= same as 'y'

integrate one side, must integrate other side

EXAMPLE 1

$$\int 3x\,(\underbrace{x^2+4}_{f(x)})^{5}\ dx$$

n

$$\frac{3}{2}\int [x^2+4]^5 \cdot 2x\quad dx = \frac{\cancel{3}^{1}}{2}\ \frac{(x^2+4)^6}{\cancel{6}\ 2} + c$$

$2\times2=4$

$$= \frac{1}{4}(x^2+4)^6 + c$$

EXAMPLE 2

b) $\int x\,(1-x^2)^7\ dx$

$= 1$

$$1\int (1-x^2)^7\ \underline{-2x}$$

f'(x)

$$-\frac{1}{2}\int (1-x^2)^7 - 2x\quad dx$$

↓ integrate

$$= -\frac{1}{2}\ \frac{(1-x^2)^8}{8} + c$$

$=$

EXAMPLE 4

3b) $\int \frac{-3x\sqrt{1-x^2}}{4}\quad dx$

$$\frac{1}{4}\int (1-x^2)^{1/2} \cdot -3x\ dx$$

$$\frac{3}{4}\int (1-x^2)^{1/2} \cdot -x\ dx$$

↑ want a 2 here

$$\frac{3}{4}\cdot\frac{1}{2}\int (1-x^2)^{1/2} \cdot -2x\quad dx$$

$$= \frac{2}{3}\,\frac{3}{8}(1-x^2)^{3/2} + c$$

$\div \frac{3}{2} = x^{2/3}$

EXAMPLE 3

$$\int 5x^2\,(3+2x^3)^4\quad dx$$

$$\int (3+2x^3)^4 \cdot (\underline{6x^2})$$

$$\frac{5}{6}\int (3+2x^3)^4\ 6x^2\ dx$$

$$= \frac{\cancel{5}}{6}\ \frac{(3+2x^3)^5}{\cancel{5}} + c$$

$$= \frac{(3+2x^3)^5}{6} + c$$

EXAMPLE 5

3c) $\int \frac{3x}{\sqrt{1+2x^2}}\quad dx$

$$\int (1+2x^2)^{-1/2} \cdot 3x \cdot dx$$

$$\frac{3}{4}\int (1+2x^2)^{-1/2}\ 4x \cdot dx$$

$$\frac{2}{1}\,\frac{3}{4}(1+2x^2)^{1/2} + c$$

$$\frac{3}{2}(1+2x^2)^{1/2} + c$$

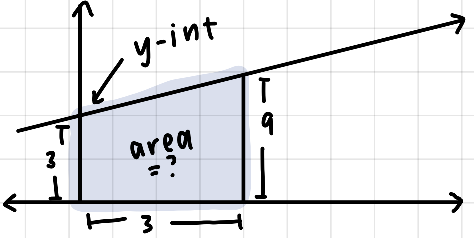
$$= \frac{1}{4}(1-x^2)^{1/2} + c$$

DEFINITE INTEGRALS

so far:

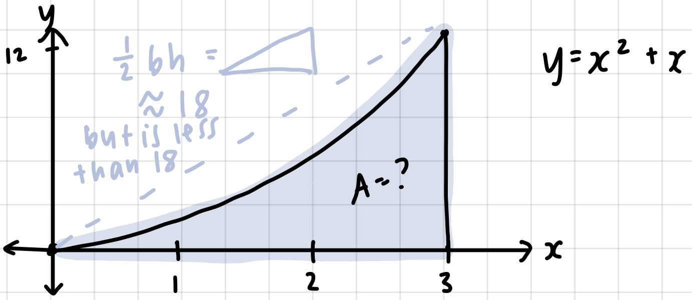
$\int x^v \, dx$ $\qquad$ $\int_a^b f(x) \, dx = \text{value}$

digression



$A = \frac{(9+3)}{2} \times 3$

$= 18 \text{ units}^2$



x	1	2	3
y	2	6	12

trapeziums

$A = 1 + 4 + 9 \quad = 14$

exact $= 13 \cdot 5$

$y' = 2x + 1 \quad = 0$

$x = -1/2$

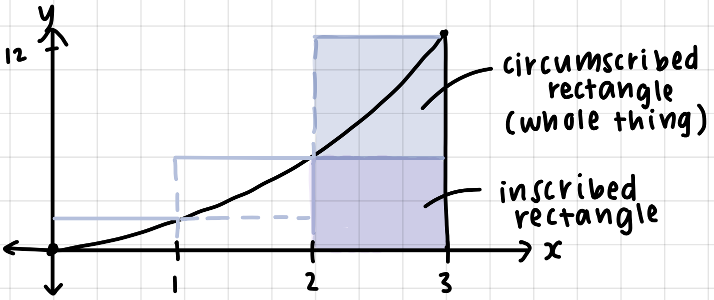
↓

rectangles

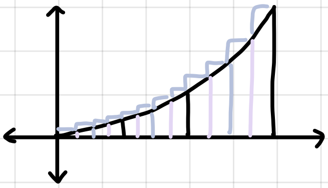
inscribed: $0 + 2 + 6 \qquad = 8$

circumscribed: $2 + 6 + 12 \qquad = 18$

$18 + 8 = 26 \qquad \div 2 = \underline{13}$



if we increase the number of rectangles, the accuracy improves



the sum of all such rectangles with width small (really small) gives us the area under the curve

the sum of the rectangles isn't shown by Σ

it's $\int_a^b$

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

F= antiderivative of f

EXAMPLE 1

$$\int_0^3 x^2 + x \, dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^3$$

for definite integrals, don't need +c (would minus each other)

$$= \frac{3^2}{3} + \frac{3^2}{2} - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)$$

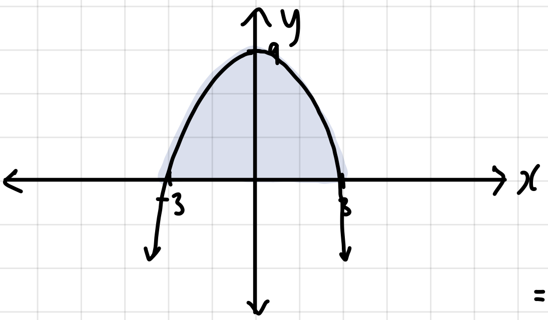
$$= 9 + \frac{9}{2}$$

$$= \frac{27}{2} \quad = 13 \cdot 5$$

EXAMPLE 2

determine the area bounded by the function $y=9-x^2$ and the x-axis

goes thru ±3



$$A = {}_{-3}\int^{3} 9-x^2\,dx$$

$$= 2\,{}_{0}\int^{3} 9-x^2\,dx$$

↖ symmetrical ∴ x2 = same as if between -3 → 3

$$= 2\left[9x-\frac{x^3}{3}\right]_0^3$$

$$= 2\left[27-9-(0)\right]$$

$$= 2[18]$$

$$A = 36$$

verify w/ ${}_{-3}\int^{3}$

$$A = \left[9x-\frac{x^3}{3}\right]_{-3}^{3}$$

$$= [27-9-(-27+9)]$$

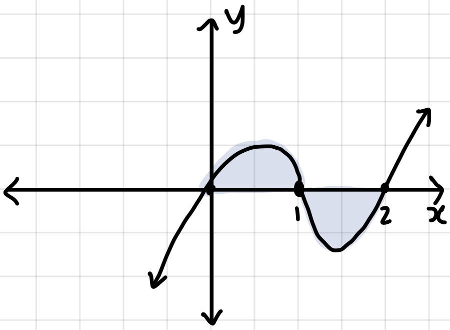
$$= [54-18]$$

$A = 36$ ∴ same

EXAMPLE 3

$y = x(x-1)(x-2)$

- cubic
- find if ∿ or ∿
  ↳ is ∿ (normal)



DOESN'T WORK - JUST TO SHOW

$$y = (x^2-x)(x-2)$$

$$= x^3-3x^2+2x$$

$$A = {}_{0}\int^{2} x^3-3x^2+2x \cdot dx$$

$$= \left[\frac{x^4}{4}-x^3+x^2\right]_0^2$$

$$= 4-8+4$$

$= 0$ ↖ area isn't 0 ∴ doesn't work (clearly not zero)

put − to oppose negative area

$$A = {}_{0}\int^{1} x^3-3x^2+2x \cdot dx \;-\; {}_{1}\int^{2} x^3-3x^2+2x \cdot dx$$

$$= \left[\frac{x^4}{4}-x^3+x^2\right]_0^1 - \left[\frac{x^4}{4}-x^3+x^2\right]^2$$

subbed 2 in (top) ↙ ↓ subbed 1 in (bottom)

$$= \left[\underbrace{\tfrac{1}{4}} \cancel{-1} \cancel{+1} \cancel{-(0)}\right] - \left[\cancel{4} \cancel{-8} \cancel{+4} - \left(\underbrace{\tfrac{1}{4}} \cancel{-1} \cancel{+1}\right)\right]$$

$$= \frac{1}{4}+\frac{1}{4} \qquad = \frac{1}{2}$$

on calculator

$${}_{\square}\int^{\square} \square\, d\square$$

$${}_{0}\int^{2} x^3-3x^2+2x\,dx$$

$= 0$ ∴ wrong

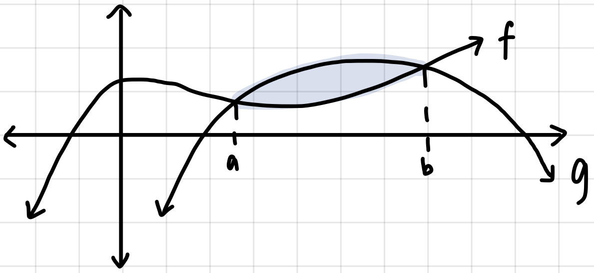
use absolute value ↙ ↙

$${}_{0}\int^{2} \left|x^3-3x^2+2x\right| dx$$

$= 0 \cdot 5$ or $\frac{1}{2}$

∴ correct

AREAS BETWEEN 2 FUNCTIONS



$A = \int_a^b g - f \, dx$

doesn't matter
what quadrant
it is in

EXAMPLE

$y = x^2$ $y = 2x$ determine area bounded by two functions

intersect when $x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\therefore \underset{a}{x=0}$ or $\underset{b}{x=2}$

$A = \int_0^2 2x - x^2 \, dx$

$= \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= 4 - \frac{8}{3}$

$= \frac{4}{3}$ units^2

if doing 2→0,
would get $-\frac{4}{3}$ but
can change the
sign

RECTILINEAR MOTION (WITH INTEGRATION)

Q10 EX3C

initially @ 0 (origin)
$t = 0, \ x = 0, \ v = 14$

$a = (3t - 11)$

need expression for velocity

$a = 3t - 11$

$v = \int 3t - 11 \, dt$

$= \frac{3t^2}{2} - 11t + c$

@ $t = 0, \ V = 14$

$\therefore c = 14$

$\therefore v = \frac{3t^2}{2} - 11t + 14$

need v, next time @ origin
$\therefore$ need displacement (= to 0)

$x = \int \frac{3t^2}{2} - 11t + 14 \, dt$

$= \frac{t^3}{2} - \frac{11t^2}{2} + 14t + c_2$

$t = 0, x = 0 \ \therefore c_2 = 0$

$\therefore x = \frac{t^3}{2} - \frac{11t^2}{2} + 14t$

next @ 0 is when $x = 0$

$0 = \frac{t^3}{2} - \frac{11t^2}{2} + 14t$

$0 = t\left(\frac{t^2}{2} - \frac{11t}{2} + 14\right)$

initially

$t = 0$ or $\frac{t^2}{2} - \frac{11t}{2} + 14 = 0$

$t^2 - 11t + 28 = 0$

$(t-4)(t-7) = 0$

$t = 4$ or $\cancel{t=7}$

looking for next time

$t = 4 \quad v = \frac{3(4)^2}{2} - 11(4) + 14$

$= 24 - 44 + 14$

$= -6 \text{ m/s}$

Q15 EX3C

maths for "fun"

$a = 3t + 2$

when $t = 0, v > 0$
in 4th sec (i.e. between 3 + 4)
travels 30m

---

then find c_1

$v = \int 3t + 2 \, dt$

$= \frac{3t^2}{2} + 2t + c_1$

$x = \frac{t^3}{2} + t^2 + c_1 t + c_2$

$x(3) = \frac{27}{2} + 9 + 3c_1 + c_2$

$x(4) = \frac{64}{2} + 16 + 4c_1 + c_2$

difference = 30

$x(4) - x(3) = 30$

$32 + 16 + 4c_1 + c_2 - \left(\frac{27}{2} + 9 + 3c + c_2\right) = 30$

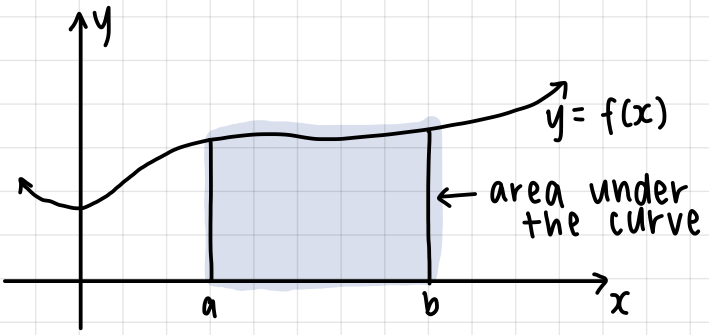
$48 + 14c_1 \cancel{+ c_2} - 13\frac{1}{2} - 9 - 3c_1 \cancel{- c_2} = 30$

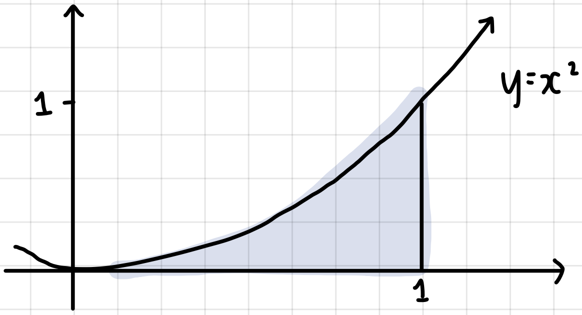
properties of $\int$

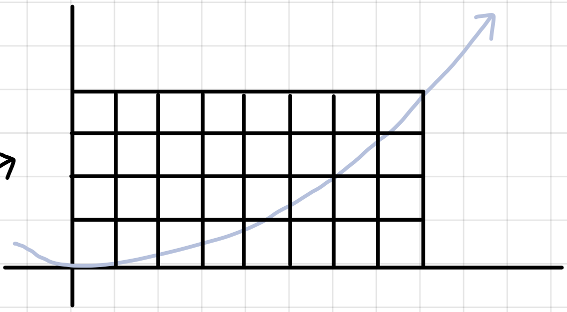
① $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$

② $\int_{a}^{c} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx$

AREA UNDER A CURVE (TEXTBOOK NOTES)





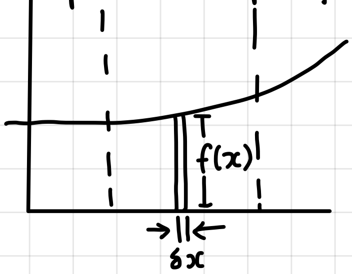


'count the squares'
≈17 squares

$$\frac{17}{50} = 0{\cdot}34 \text{ units}^2$$

'area of the strips'
i.e circumscribe & inscribe
overestimate & underestimate

to find area under $f(x)$ from $x = a$ to $x = b$
↳ split up into strips of δx



$$= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$$

writes as:

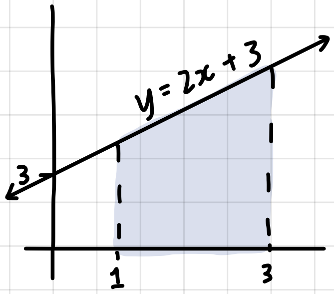
$$\int_a^b f(x)\,dx$$

STEPS TO EVALUATE

$$\int_a^b f(x)\,dx$$

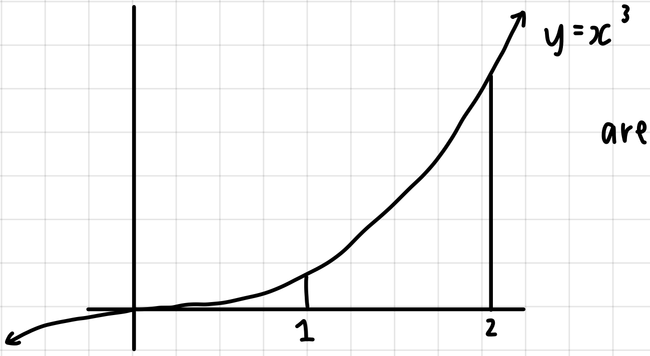
① antidifferentiate $f(x)$ w/ respect to x (and omit the $+c$)
② sub b into answer from ①
③ sub a into answer from ①
④ calculate: (② - ③)
i.e. $\int_a^b f'(x)dx = f(b)-f(a)$

EXAMPLE



area of trapezium $= 2\left(\frac{9+5}{2}\right)$
$= 14 \text{ units}^2$

calculus
area $= \int_1^3 (2x+3)\,dx$
$= \left[x^2+3x\right]_1^3$
$= (3^2+3(3)) - (1^2+3(1))$
$= 14 \text{ units}^2$



$$\text{area} = \int_1^2 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]^2$$

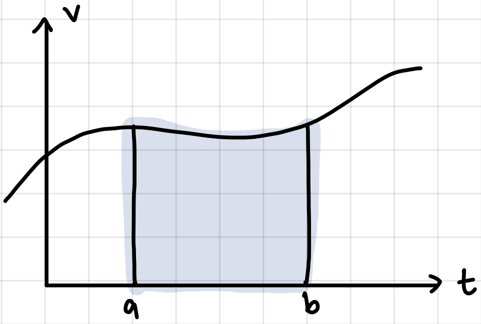
$$= \frac{2^4}{4} - \frac{1^4}{4}$$

$$= 3{\cdot}75 \text{ units}^2$$

LIMIT AS δx TENDS TO ZERO

$$\sum_{x=a}^{x=b} f(x)\,\delta x$$

DISPLACEMENT AS AREA UNDER CURVE



displacement $= \int_a^b v(t)\,dt$

DIFFERENTIATION (EXPONENTIAL)

$y = e^{x}$

↑ 2.718

$$\boxed{\text{if } y = e^{x} \quad \frac{dy}{dx} = e^{x}}$$

$y = e^{f(x)}$

let $u = f(x)$

$\frac{du}{dx} = f'(x)$

$\therefore y = e^{u}$

$\frac{dy}{du} = e^{u}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= e^{u} \cdot f'(x)$

$$\therefore \boxed{\frac{dy}{dx} = e^{f(x)} \cdot f'(x)}$$

if $y = e^{\text{thing}}$

$y' = e^{\text{thing}}$ × derivative (thing)

EXAMPLE 1

$y = e^{x^2+2x}$

$y' = e^{x^2+2x} \cdot (2x+2)$

EXAMPLE 2

$y = (x^2)(e^{2x})$

$y' = 2x e^{2x} + x^2 \cdot e^{2x}(2)$

↑ product rule

EXAMPLE 3

$y = \frac{x^2}{e^{2x}}$

$y' = \frac{e^{2x} \cdot 2x - x^2 \; 2e^{2x}}{(e^{2x})^2}$

↑ quotient rule

$\frac{2xe^{2x}(1-x)}{e^{4x}}$

$= \frac{2x(1-x)}{e^{2x}}$

can be x^2e^{-2x}

$y' = 2xe^{-2x} + x^2 \cdot e^{-2x}(-2)$

$= \frac{2x}{e^{2x}} - \frac{2x^2}{e^{2x}}$

$= \frac{2x - 2x^2}{e^{2x}}$

$= \frac{2x(1-x)}{e^{2x}}$

EXAMPLE - 6B U3

Q33 $y = 5e^{2x}$ @ $(0,5)$

$y' = 5e^{2x} \cdot 2$

$= 10e^{2x}$

$y'\big|_{x=0} = 10(1)$

$= 10$

equation

$y - 5 = 10(x-0)$

$y = 10x + 5$

$$\boxed{y - y_1 = m(x - x_1)}$$

INTEGRATION (EXPONENTIAL)

if $\frac{dy}{dx} = e^x$

$y = \int e^x \, dx$

$= e^x + c$

$\int e^{f(x)} f'(x) \, dx$

$= e^{f(x)} + c$

EXAMPLE 1

$\frac{1}{3} \int e^{3x} 3 \, dx$

$= \frac{1}{3} e^{3x} + c$

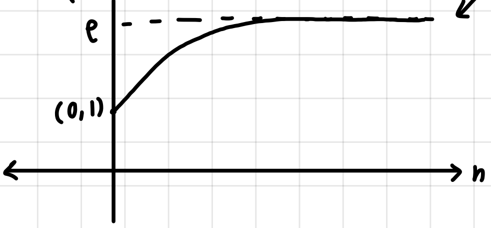
EXAMPLE 2

$\frac{1}{2} \int e^{x^2} 2x \, dx$

$= \frac{1}{2} e^{x^2} + c$

variable part
has to be in
the integral
(can fiddle
around w/
the constants)

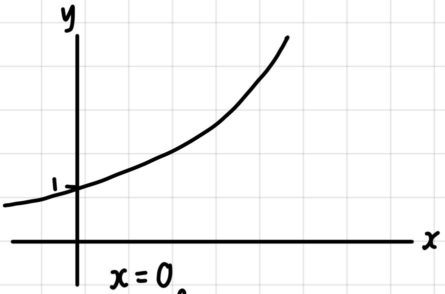
E = $(1+\frac{1}{n})^n$



tends toward
2·71827....

$y = e^x$

$\frac{dy}{dx} = e^x$



$x = 0$

$y = e^0 = 1$

$(0,1)$

EXPONENTIAL FUNCTION (TEXTBOOK NOTES)

2·71828

if the rate was 7% per annum
w/ continuous compounding:

$$\$P \times (2{\cdot}71828)^{0{\cdot}07}$$

after 2 years:

$$\$P \times (2{\cdot}71828^2)^{0{\cdot}07}$$

this number of 2·71828 came from consideration of: $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]$

↖ this is equal to e

$\therefore e^1 \approx 2{\cdot}71828$

GROWTH + DECAY

$$\underset{\substack{\text{amount}\\\text{present @}\\\text{time } t}}{A} = \underset{\substack{\text{initial}\\\text{amount}}}{A_0} e^{\overset{\text{constant}}{k}t}$$

INTEGRATING EXPONENTIAL FUNCTIONS

if $y = e^x$ then $\frac{dy}{dx} = e^x$

thus $\int e^x dx = e^x + c$

also if $y = e^{f(x)}$, we let $u = f(x)$
and so $y = e^u$

then by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times f'(x)$$
$$= f'(x)e^{f(x)}$$

$$\int f'(x)e^{f(x)} dx$$
$$= e^{f(x)} + c$$

if $y = e^x$ then $\frac{dy}{dx} = e^x$

if $y = e^{f(x)}$ then by the chain rule $\frac{dy}{dx} = f'(x)e^{f(x)}$

MORE ON GROWTH + DECAY

if Ae^{kt} then $\frac{dy}{dt} = kAe^{kt}$

i.e $\frac{dy}{dt} = ky$

in functions of the form $y = Ae^{kt}$, the rate of change of y with respect to t is proportional to y itself

↖ describes why functions in the form $y = Ae^{kt}$ describe growth or decay

any growth or decay situation in which population proportional to the population itself

i.e. $\frac{dP}{dt} = kP$ can be shown by an equation: $P = P_0 e^{kt}$

where P_0 is the population @ $t=0$

in terms of x and y

if $\frac{dy}{dx} = ky$ then $y = y_0 e^{kx}$ where y_0 is the value of y when $x=0$

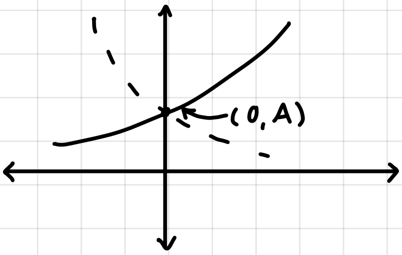
GROWTH + DECAY USING e FUNCTION

for Ex 6C

let $y = Ae^{kt}$

then $\frac{dy}{dt} = Ae^{kt} \cdot k$

$\therefore \frac{dy}{dt} = kAe^{kt}$



sub $y = Ae^{kt}$

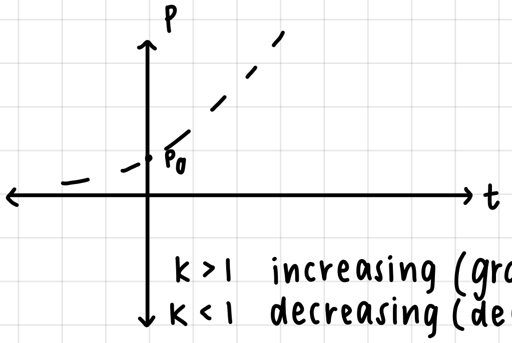
so $\frac{dy}{dt} = ky$

$$\boxed{\frac{dy}{dt} \propto y \quad \text{then } \frac{dy}{dt} = ky}$$

for any function of the form $y = Ae^{kt}$, the rate of change with respect to t is proportional to itself

hence, if $\frac{dP}{dt} = kP$ then $P = P_0 e^{kt}$

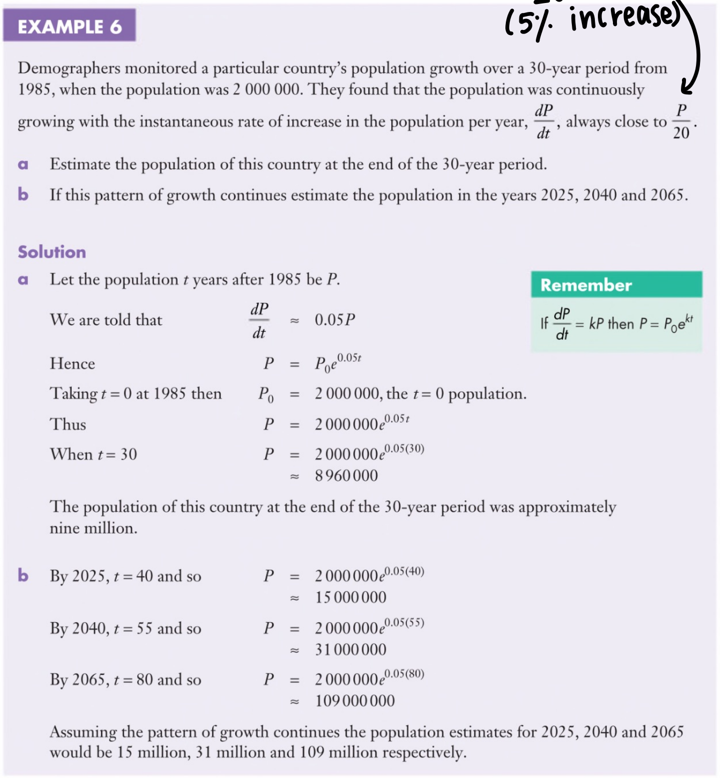
↖ initial value (t = 0)
(P-intercept)



k > 1 increasing (growth)
k < 1 decreasing (decay)

EXAMPLES FROM U3 TEXTBOOK pg 128-129

$\frac{1}{20} = 0.05$
(5% increase)



EXAMPLE 6

Demographers monitored a particular country's population growth over a 30-year period from 1985, when the population was 2 000 000. They found that the population was continuously growing with the instantaneous rate of increase in the population per year, $\frac{dP}{dt}$, always close to $\frac{P}{20}$.

a Estimate the population of this country at the end of the 30-year period.

b If this pattern of growth continues estimate the population in the years 2025, 2040 and 2065.

Solution

a Let the population t years after 1985 be P.

We are told that $\frac{dP}{dt} \approx 0.05P$

Hence $P = P_0 e^{0.05t}$

Taking $t = 0$ at 1985 then $P_0 = 2\,000\,000$, the $t = 0$ population.

Thus $P = 2\,000\,000e^{0.05t}$

When $t = 30$ $P = 2\,000\,000e^{0.05(30)}$

$\approx 8\,960\,000$

The population of this country at the end of the 30-year period was approximately nine million.

Remember

If $\frac{dP}{dt} = kP$ then $P = P_0e^{kt}$

b By 2025, $t = 40$ and so $P = 2\,000\,000e^{0.05(40)}$

$\approx 15\,000\,000$

By 2040, $t = 55$ and so $P = 2\,000\,000e^{0.05(55)}$

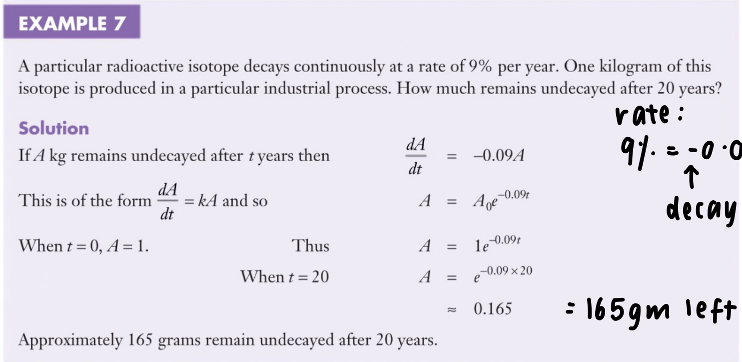
$\approx 31\,000\,000$

By 2065, $t = 80$ and so $P = 2\,000\,000e^{0.05(80)}$

$\approx 109\,000\,000$

Assuming the pattern of growth continues the population estimates for 2025, 2040 and 2065 would be 15 million, 31 million and 109 million respectively.

If the situation involves a quantity **decaying** rather than growing then the rate of change of the quantity with respect to time will be negative, rather than positive. (See the next example.)



EXAMPLE 7

A particular radioactive isotope decays continuously at a rate of 9% per year. One kilogram of this isotope is produced in a particular industrial process. How much remains undecayed after 20 years?

Solution

If A kg remains undecayed after t years then $\frac{dA}{dt} = -0.09A$

This is of the form $\frac{dA}{dt} = kA$ and so $A = A_0e^{-0.09t}$

When $t = 0$, $A = 1$. Thus $A = 1e^{-0.09t}$

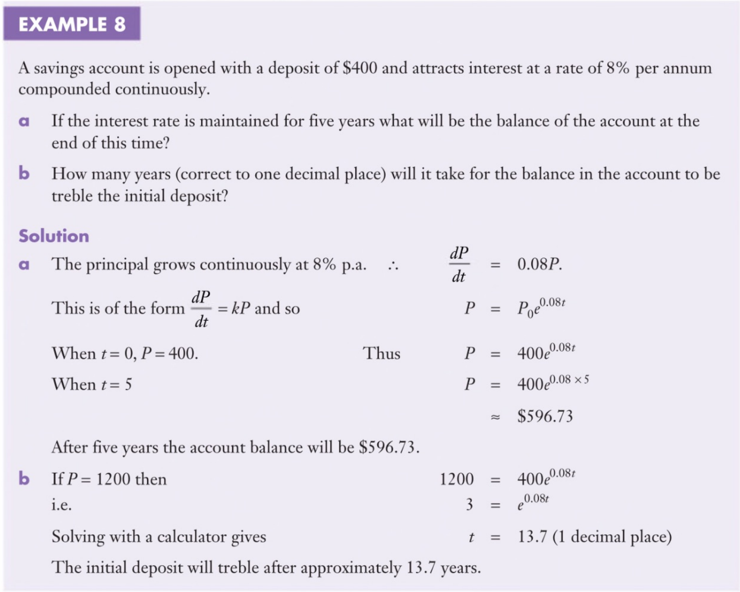
When $t = 20$ $A = e^{-0.09 \times 20}$

≈ 0.165

Approximately 165 grams remain undecayed after 20 years.

rate:
9% = -0.09
↑
decaying

= 165gm left



EXAMPLE 8

A savings account is opened with a deposit of \$400 and attracts interest at a rate of 8% per annum compounded continuously.

a If the interest rate is maintained for five years what will be the balance of the account at the end of this time?

b How many years (correct to one decimal place) will it take for the balance in the account to be treble the initial deposit?

Solution

a The principal grows continuously at 8% p.a. $\therefore$ $\frac{dP}{dt} = 0.08P$.

This is of the form $\frac{dP}{dt} = kP$ and so $P = P_0e^{0.08t}$

When $t = 0$, $P = 400$. Thus $P = 400e^{0.08t}$

When $t = 5$ $P = 400e^{0.08 \times 5}$

$\approx \$596.73$

After five years the account balance will be \$596.73.

b If $P = 1200$ then $1200 = 400e^{0.08t}$

i.e. $3 = e^{0.08t}$

Solving with a calculator gives $t = 13.7$ (1 decimal place)

The initial deposit will treble after approximately 13.7 years.

TRIG

basics:

if $y = \sin x$
$y' = \cos x$

if $y = \cos x$
$y' = -\sin x$

$\therefore$ if $y = \frac{\sin}{\cos}(\text{thing})$

then $y' = \frac{\cos}{-\sin}(\text{thing}) \cdot (\text{derivative of thing})$

what if:

$y = \sin(3x)$
$y' = 3\cos(3x)$

let $u = 3x$
& $y = \sin u$

chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \cos u \cdot 3$$
$$= \cos(3x) \cdot 3$$
$$= 3\cos(3x)$$

sub $u = 3x$ back in

EXAMPLE

$y = \cos(-3x^2)$
$y' = -\sin(-3x^2) \cdot (-6x)$
$= 6x\sin(-3x^2)$

derivative of $(-3x^2)$

EXAMPLE 2

$y = \sin(2e^{2x})$
$y' = \cos(2e^{2x}) \cdot (4e^{2x})$

$(2e^{2x} \cdot 2)$ $\frac{dy}{dx} =$

EXAMPLE 3

$y = e^{2x}\sin(3x)$
use product rule

$$y' = 2e^{2x} \cdot \sin(3x) + e^{2x} \cdot \overbrace{\cos(3x) \cdot 3}^{y' \text{ of } \sin(3x)}$$

EXAMPLE 4

$$y = \frac{2e^{x^2}}{\cos(x^2)}$$

use quotient rule

$$y' = \frac{\cos(x^2) \cdot 2e^{x^2} \cdot 2x - 2e^{x^2} \cdot (-\sin(x^2)) \cdot (2x)}{(\cos(x^2))^2}$$

$\cos x \times \cos x$
$= \cos^2 x$

INTEGRATION OF TRIG FUNCTIONS

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

DOESN'T WORK - DON'T DO THIS

$$\cancel{\int \cos(x^2)\,dx \neq \frac{\sin(x^2)}{2x}} + c$$

EXAMPLE

$$\int \cos 2x \, dx$$
$$= \frac{1}{2}\sin 2x + c$$

in general:

$$\int \cos(ax)\,dx$$
$$= \frac{\sin(ax)}{a} + c$$
$$= \frac{1}{a}\sin(ax) + c$$

this has to be linear

recall:

$$\int [f(n)]^n \cdot f'(x)\,dx$$
$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

EXAMPLE

$$\int \sin^2 x \cdot \cos x \, dx$$

RECALL

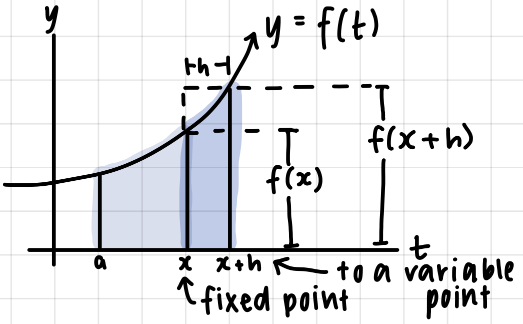
$$\int [f(n)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

EXAMPLE

$$\int \sin^2 x \cdot \cos x \, dx = \frac{\sin^3 x}{3} + c$$

$\sin^2 x$
$= (\sin x)^2$

FUNDAMENTAL THEOREM



▨ $= A(x+h) - A(x)$

[from a to x+h] - [from a to x] = [area from x → x+h]

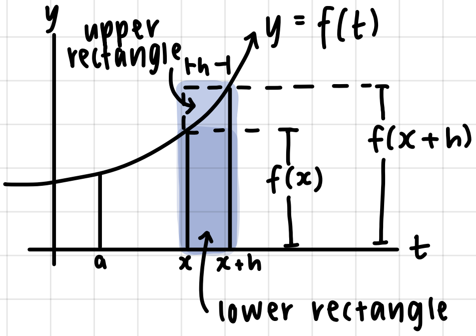
DEFINE

$A(x) = {}_a\int^x f(t)\,dt$

UPPER + LOWER RECTANGLES

$h\cdot f(x) < A(x+h) - A(x) < h\cdot f(x+h)$

lower upper



then ÷ h

$= f(x) < \frac{A(x+h) - A(x)}{h} < f(x+h)$

↓ definition of a derivative

THEN LIMIT

as $h \to 0$

$\underbrace{f(x) < A'(x) < f(x)}_{\text{same}}$

called the 'squeeze theorem'

same ∴ implies that

$A'(x) = f(x)$

i.e. $\frac{d}{dx}\,{}_a\int^x f(t)\,dt = f(x)$

↑ derivative (same as $A'(x)$)

RESULT OF FIRST FUNDAMENTAL THEOREM

NOW INTEGRATE

$A'(x) = f(x)$

$\int A(x) = F(x) + c$ [where $F(x) = \int f(x)$]

↑ with respect to x

at $t = a$ ↙ beginning point

$A(a) = F(a) + c$

$0 = F(a) + c$

$\therefore c = -F(a)$

$\therefore A(x) = F(x) - F(a)$

at $t = b$ ↗ 'other end'

$A(b) = F(b) - F(a)$

$_a\int^b f(t)\,dt = F(b) - F(a)$ ↗ definition for a definite integral

↖ from definition above

RESULT OF SECOND FUNDAMENTAL THEOREM

EXAMPLE

$\frac{d}{dx}\,{}_2\int^x t^2 + 3\,dt$

$= x^2 + 3$

EXAMPLE 2

$\frac{d}{dx}\,{}_5\int^x e^{t^2}\,dt$

$= e^{x^2}$

EXAMPLE 3

$F(x) = {}_5\int^x e^t + 5\,dt$

find max.

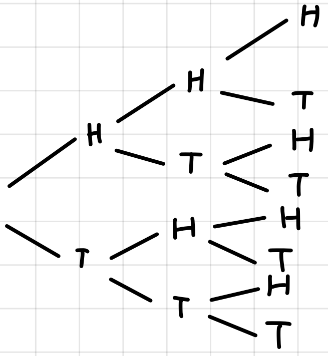
$F'(x) = e^x + 5$

set = to 0

DRV'S

DRV = discrete random variable
discrete = we count (integer)
random = no bias
variable = must have more than
1 outcome

consider tossing 3 fair coins



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

suppose we are interested in the variable # of heads obtained in 3 tosses

8 combos   0, 1, 2, 3 heads   (X)

x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

P(X=0) P(X=1) P(X=2) P(X=3)

CONDITION FOR DRV'S

- outcomes (x) must be discrete
- outcomes (x) must be random
- $P(X=x) \geq 0$
- $\Sigma P(X=x) = 1$

OR A cumulative distribution table

x	0	1	2	3
P(X≤x)	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

P(x=0)
+ P(x=1)

WHICH ARE DRV'S ?

①

x	0	1	3	5
P(X=x)	-0·1	0·1	0·4	0·5

✗

②

x	0	1	3	5
P(X=x)	0·1	0·1	0·4	0·4

Σ=1 ✓

③

x	-3	-2	1
P(X=x)	0·6	0·3	0·1

Σ=1 ✓

DRV'S BY RULES

$$P(X=x) = \begin{cases} k(4-x), & x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

x	1	2	3
P(X=x)	3k	2k	k

from plugging x into k(4-x)

Σ=1

① DRV? yes ∴ Σ=1
② ∴ 6k = 1, find k:
$k = \frac{1}{6}$
③ fill out table using k
$\frac{3}{6}$ $\frac{2}{6}$ $\frac{1}{6}$

a) $P(X \text{ is odd}) = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$
b) $P(X \geq 2) = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
c) $P(X=3 \overset{\swarrow \text{given that}}{|} X \geq 2) = 1/6 \Big/ 2/6 + 1/6 = \frac{1/6}{1/2}$
$= \frac{1}{6} \times 2 = \frac{2}{6} = \frac{1}{3}$

EXPECTED VALUE + VARIANCE

counting the # of birds on a fence in a day

x	f	$P(X=x)$
1	3	3/20
2	7	7/20
3	5	5/20
4	1	1/20
5	4	4/20
adds to:	20	1

average no. of birds:

$$\frac{(1\times3)+(2\times7)+(3\times5)+(4\times1)+(5\times4)}{20} = 2.8 \text{ birds}$$

$$\bar{x} = E(x) = \left(1\times\frac{3}{20}\right)+\left(2\times\frac{7}{20}\right)+\left(3\times\frac{5}{20}\right)+\left(4\times\frac{1}{20}\right)+\left(5\times\frac{4}{20}\right)$$

$$= 2.8$$

$$= \sum x \cdot P(x) = \mu$$

↑ mean

↑ expected value of x

also: $\text{var}(X) = \sum (x-\mu)^2 \, P(x)$

x	$(x-\mu)^2$	$P(X=x)$
1	$(1-2.8)^2$	3/20
2	$(2-2.8)^2$	7/20
3	$(3-2.8)^2$	5/20
4	$(4-2.8)^2$	1/20
5	$(5-2.8)^2$	4/20

$$\text{var}(x) = \left(3.24\times\frac{3}{20}\right)+\left(0.64\times\frac{7}{20}\right)+\left(0.04\times\frac{5}{20}\right)+\left(1.44\times\frac{1}{20}\right)+\left(4.84\times\frac{4}{20}\right)$$

$$= 1.76$$

= standard deviation^2

$$\text{s.d.} = \sigma = \sqrt{1.76}$$

$$= 1.33$$

$$\text{var}(X) = E(X^2) - [E(X)]^2$$

x	$P(X=x)$
1	3/20
2	7/20
3	5/20
4	1/20
5	4/20

$$\text{var}(X) = \left(1^2\times\frac{3}{20}\right)+\left(2^2\times\frac{7}{20}\right)+\left(3^2\times\frac{5}{20}\right)+\left(4^2\times\frac{1}{20}\right)+\left(5^2\times\frac{4}{20}\right) - 2.8^2$$

$$= (0.15 + 1.4 + 2.25 + 0.8 + 5) - 2.8^2$$

$$= 1.76$$

s.d. ↘

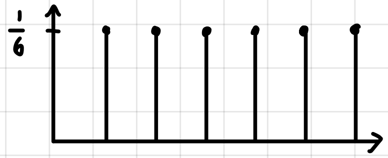
$$\therefore \sigma = \sqrt{1.76} \quad = 1.33$$

DISCRETE UNIFORM DISTRIBUTION

e.g. rolling a fair die. X-value of die face

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

graphically:



$E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$

$= 3.5$

$\text{var}(X) = \Sigma(x-\mu)^2 x\ P(x)$

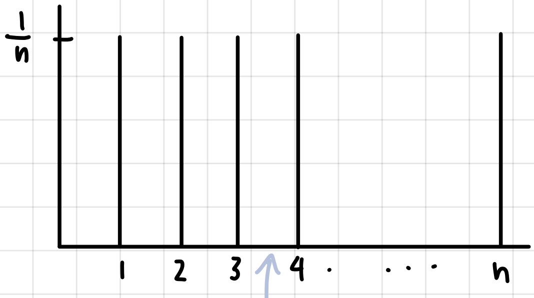
$= 2.916$

graph + table (table)

list 1	list 2
1	1.6666
2	1.6666
3	1.6666
4	1.6666
5	1.6666
6	1.6666

σx = standard deviation

in general



3.5 same as median

$$E(X) = \sum_{x=1}^{n} x \times \frac{1}{n}$$

$$= \frac{n+1}{2}$$

on calc: $\sum_{\square=\square}^{\square}(\square)$

$\therefore \sum_{x=1}^{n}\left(x\times\frac{1}{n}\right)$

$= \frac{n+1}{2}$

mean

$$\text{var}(X) = E(X^2) - \mu^2$$

$$= \sum_{x=1}^{n} x^2 \times \frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

hit the 'simp' button to simplify

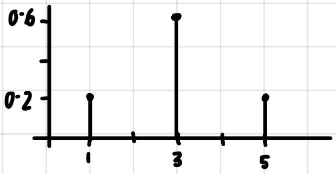
$$= \frac{n^2-1}{12}$$

standard deviation is $\sqrt{\ }$ of variance

LINEAR CHANGES

consider

x	1	3	5
$P(X=x)$	0·2	0·6	0·2

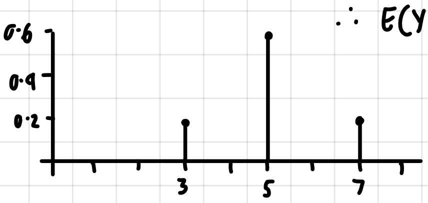


$E(X) = 3$] from calc
$\sigma = 1{\cdot}2649$

if $Y = X + 2$ (change of origin)

y	3	5	7
$P(Y=y)$	0·2	0·6	0·2

$E(Y) = 5$
$\sigma = 1{\cdot}2649$
↑ 'spread' (s·d·)



symmetry
$\therefore E(Y) = 5$

if $W = 2X$ (change of scale)

w	2	6	10
$P(W=w)$	0·2	0·6	0·2

$E(W) = 6$ (original $\sigma \times 2$)
$\sigma = 2(1{\cdot}2649)$
$= 2{\cdot}5298$

summary:

data	mean	s·d·
x	μ	σ
$x + c$	$\mu + c$	σ
kx	$k\mu$	$\lvert k\rvert \times \sigma$
$kx + c$	$k\mu + c$	$\lvert k\rvert \times \sigma$

↑ absolute value
(cannot have $-$s·d·)

BINOMIAL DISTRIBUTION (DISCRETE)

Bernoulli trial

- a single event/outcome and it is either a success or failure (2 outcomes)
- probability of success must remain constant

EXAMPLE

tossing a coin once X : a head

x	0	1
P(X=x)	0·5	0·5

$E(x) = (0\times0{\cdot}5)+(1\times0{\cdot}5)$

$= 0{\cdot}5$

$var(X) = 0\times0{\cdot}5 + 1^2\times0{\cdot}5 - (0{\cdot}5)^2$

$= 0{\cdot}25$

$s.d. = 0{\cdot}5$

in general:

if $P(success) = P$

then $P(failure) = 1-P$

$E(x) = P$

$var(X) = P(1-P)$

now toss coin 3 times:

(repeated Bernoulli trials)

*independent events

x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0$

$\binom{3}{0}\overbrace{\left(\frac{1}{2}\right)^0}^{P(tail)}\left(\frac{1}{2}\right)^3$

$\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$

$\binom{3}{1}\left(\frac{1}{2}\right)^1\overbrace{\left(\frac{1}{2}\right)^2}^{P(head)}$

HHH
HHT
HTH
HTT
TTT
THT
THH
TTH

now repeating a Bernoulli trial n times with a probability of success P[failure (1-P)] is called a Binomial distribution

i.e. $P(X=x) = \binom{n}{x}p^x(1-p)^{n-x}$

$E(X) = \mu = nP$

$var(X) = np(1-P)$

notation:

$X \sim Bin(\underset{\uparrow \text{ no. of trials}}{n}, \underset{\text{probability of success}}{p})$

EXAMPLE ON ONENOTE

$X \sim Bin(n, 0{\cdot}6)$

i.e. $Bin(10, 0{\cdot}6)$

a) $P(X>6) = P(X\geq7)$

b) $\mu = nP$

$var(X) = nP(1-P)$

$\left(\frac{4\sqrt{15}}{5}\right)^2 = n(0{\cdot}6)(0{\cdot}4)$

$n = 40$

$\therefore \mu = 40(0{\cdot}6)$

$= 24$

c) $\binom{n}{n}(0{\cdot}6)^n(0{\cdot}4)^0 = 0{\cdot}01$

$0{\cdot}6^n = 0{\cdot}01$

$= 9{\cdot}015$

≈ 9

ON CALC

stats

calc → distribution

binomial PD or CD

PD ↑ single event

CD ↑ cumulative (more than 1)

lower : 7
upper · 10
numtrial : 10
pos : 0·6

pos: probability of success

then hit next
→ gives probability

LOGARITHMS

if $2^x = 16$
$2^x = 2^4$
$x = 4$
$4 = \log_2 16$

what if $2^x = 15$?
x must lie between 3 and 4
by calc: $x \approx 3{\cdot}91$ (2d.p)

on classpad keyboard: $\log_{\blacksquare}\square$
$\log_{10}(\square)$

scientific:
log → base $10^{\square}$
ln → base $e^{\square}$

we say that x is the logarithm to the base 2 of 15

$\underset{\uparrow \text{ base}}{2}^{x} = 15$ i.e. $x = \log_{\underset{\uparrow \text{ base}}{2}} 15$

i.e. approx $3{\cdot}91 = \log_2 15$

i.e. 4 is the number 2 must be raised to, to give 16

positive $(a>0)$

if $a^x = b$ then $x = \log_a b$

indicial form

on formula sheet

logarithmic form

COMMON LOGS

- base 10

notation: $\log_{10} x$
$= \log x$

$\log_{10} 100$
$= 2 \quad (10^2 = 100)$

$\log_{10} \frac{1}{10}$
$= -1 \quad (10^{-1} = \frac{1}{10})$

LAWS OF LOGS

a= base

① $\log_a mn = \log_a m + \log_a n$
② $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
③ $\log_a (m^k) = k \log_a m$

EXAMPLES

indicial form: $d^n = f$
logarithmic : $n = \log_d f$

logarithmic form: $\frac{4}{3} = \log_{27} 81$
indicial : $27^{4/3} = 81$

write in the form $\log m$ ← base 10 if just log

① $\log 8 + \log 3 = \log 24$ ← 8×3
② $3\log 2 = \log 2^3 = \log 8$
③ $2 + \log 3 = \log 100 + \log 3$
$10^{②} = 100$
$= \log 300$

④ $1 - \log 2 = \log 10 - \log 2$ ← $10^1 = 100$
$= \log \frac{10}{2}$
$= \log 5$

⑤ $\log 2 + \log 4 + \log 6 = \log 48$ ← $2 \times 4 \times 6$

⑥ $\log 125 + \log 32 - \log \frac{2}{5}$
$= \log\left(\frac{125 \times \overset{16}{\cancel{32}} \times 5}{\cancel{2}}\right)$
$= \log(125 \times 16 \times 5)$
$= \log(10000)$
$= 4$ ← $10^4 = 10{,}000$

⑦ $\frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3\cancel{\log 2}}{\cancel{\log 2}} = 3$ ← from rule

DON'T $\frac{\overset{4}{\cancel{8}}}{\cancel{2}}$

⑧ $\frac{\log 27}{\log 9} = \frac{\log 3^3}{\log 3^2} = \frac{3\cancel{\log 3}}{2\cancel{\log 3}} = \frac{3}{2}$ ← 3rd law

LOG EQUATIONS

example

$\log_2 y^4 + \log_2 y = 10$

$4\log_2 y + \log_2 y = 10$

$5\log_2 y = 10$

$\log_2 y = 2$

$\uparrow\ 2^2 = 4$

$\therefore y = 4$

or can do indicial form:

$2^2 = y \quad \therefore y = 4$

example (solve for n)

$\log_2(\sqrt{2x^3}) + 1 = 4.5$

$\log_2(\sqrt{2x^3}) = 3.5$

convert to indicial:

$2^{3.5} = \sqrt{2x^3}$

$2^{7/2} = (2x^3)^{1/2}$

square both sides:

$(2^{7/2})^2 = [(2x^3)^{1/2}]^2$

$2^7 = 2x^3$

$2^7/2 = x^3$

$2^6 = x^3$

$(2^6)^{1/3} = (x^3)^{1/3}$

$\uparrow$ cube root

$2^2 = x$

$x = 4$

example:

$\log_4(x+3) + \log_4(x-3) = 2$

$\log_4[(x+3)(x-3)]$ log → indicial form $4^2 = 16$

$(x+3)(x-3) = 16$

$x^2 - 9 = 16$

$x^2 = 25$

$x = \pm 5$

$\uparrow$ cannot have negative

$\therefore x = 5$

example:

$(7^x)^2 - 4(7^x) = 0$ let $y = 7^x$

$7^x(7^x - 4) = 0$

$7^x = 0$ or $7^x = 4$

no solution $\quad 7^x = 4$

$$x = \frac{\log 4}{\log 7}$$

or $x = \log_7 4$

example

$4^{2x-8} = 70$ ↗ convert to log form

$\log_4 70 = 2x - 8$

$\log_4 70 + 8 = 2x$

$$x = \frac{\log_4 70 + 8}{2}$$

$= 4 + \frac{1}{2}\log_4 70$

example: (exact values)

$5^{5x} = 3^{2x+3}$

take logs of both sides

$\log 5^{5x} = \log 3^{2x+3}$

use log laws (3rd law)

$5x\log 5 = (2x+3)\log 3$ ↙ expand brackets

$= 2x\log 3 + 3\log 3$

$5x\log 5 - 2x\log 3 = 3\log 3$

$\uparrow$ group x's on one side

$x(5\log 5 - 2\log 3) = 3\log 3$

$\uparrow$ factor out the x

$$x = \frac{3\log 3}{5\log 5 - 2\log 3}$$ ← this is the exact value

example: $(5^x)^2 + 2(5^x) - 24 = 0$

let $y = 5^x$, $y^2 + 2y - 24 = 0$ (quadratic)

$(y-4)(y+6) = 0$

$y = 4$ or $y = -6$

finding x: $5^x = 4$ or $5^x = -6$

$\log 5^x = \log 4$ or $\cancel{\log 5^x = \log(-6)}$

$\uparrow$ can't have log of a negative no.

$x\log 5 = \log 4$

$$x = \frac{\log 4}{\log 5}$$

NATURAL LOGS

$[\log_e x] : [\ln x]$

base 'e'   L means log base e

follow all log laws + properties

GRAPHS OF LOG FUNCTIONS

$y = \log_a x$

$x > 0$

x-int $(1,0)$

$a^y = x$

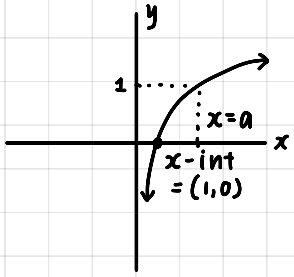
$a^0 = x$

$1 = x$

or $\log_a 1 = 0$

when $x = a$, $y = 1$ ← passes through this point

using $\log_a a^1 = 1$



note: the base doesn't matter

1. x intercept
2. when $y = 1$, $x = ?$
3. asymptote

vertical asymptote $(x = 0)$

↑ the y axis as the function is undefined at this point

TRANSFORMATIONS

of the base graph

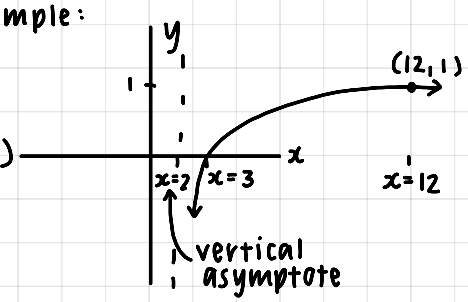
example:

$y = \log_a(x - c)$ = horizontal translation

negative = moves right

positive = moves left

$y = \log(x-2)$



$y = \log_a x + d$ = vertical translation

x-int when $y = 0$

$0 = \log_{10} x + 1$

$-1 = \log_{10} x$

$10^{-1} = x$

$1/10 = x$ $x = 0 \cdot 1$

x when $y = 1$

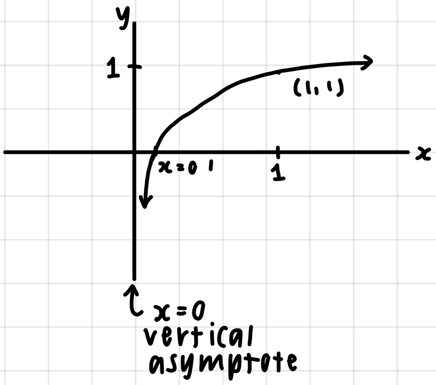
$\log_{10} 1 = 0$

$\therefore 1$

because

$\log_{10} 10^0 = 0$

$y = \log x + 1$



$y = \log_a(x-c) + d$ = combination

x-int, $y = 0$

$0 = \log(x-2) + 1$

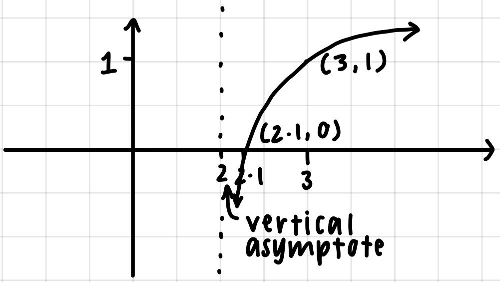
$-1 = \log(x-2)$

$10^{-1} = x - 2$ ← change to indicial

$x = 2 + 0 \cdot 1$

$= 2 \cdot 1$

$y = \log(x-2)+1$



when $y = 1$, $x = ?$

$1 = \log(x-2) + 1$

$1 = \log(\underbrace{3 - 2}_{\text{find the value of } x \text{ that makes this } = 0}) + 1$

↑ makes = 0 since $\log 1 = 0$

CALCULUS OF LOG FUNCTION

can only differentiate natural logs

$y = \ln x$

$\uparrow\ y = \log_e x$

$e^y = x$

$\boxed{x = e^y}$

$\frac{dx}{dy} = e^y$

$\therefore \frac{dy}{dx} = \frac{1}{e^y}$

$\therefore \frac{dy}{dx} = \frac{1}{x}$

example: if $y = \ln 2x$, $\frac{dy}{dx} = ??$

$\uparrow$ product

$y = \overset{=0}{\ln 2} + \ln x$

$y' = \frac{1}{x}$

example: if $y = \ln\left(\frac{x}{2}\right)$

$y = \ln x - \ln 2$

$y = \ln x - 0$

$y' = \frac{1}{x}$

example: if $y = \ln x^2$

$y = 2\ln x$

$y' = 2\left(\frac{1}{x}\right)$

$= \frac{2}{x}$

example: if $y = \ln(\sin x)$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$y = \ln u$

$\frac{dy}{du} = \frac{1}{u}$

$\therefore$ by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \cos x$$

$$= \frac{1}{\sin x} \cdot \cos x$$

$$= \frac{\cos x}{\sin x}$$

> shortcut: if $y = \ln[f(x)]$
>
> $\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

example: $y = \ln(2\cos 2x)$

$$\frac{dy}{dx} = \frac{1}{2\cos 2x} \cdot -4\sin 2x$$

$$= \frac{-4\sin 2x}{2\cos 2x}$$

example: if $y = \ln(x^2 \cdot \sin x)$

$$\frac{dy}{dx} = \frac{2x\sin x + x^2\cos x}{x^2 \cdot \sin x}$$

$\swarrow$ product rule

or

$y = \ln x^2 + \ln(\sin x)$

$= 2\ln x + \ln(\sin x)$

$y' = \frac{2}{x} + \frac{\cos x}{\sin x}$

example: $y = \ln\left(\frac{x^2-4}{x+3}\right)$

$y = \ln(x^2-4) - \ln(x+3)$

$= \ln(x+2)(x-2) - \ln(x+3)$

$= \ln(x+2) + \ln(x-2) - \ln(x-3)$

$= \frac{1}{x+2} \cdot 1 + \frac{1}{x-2} \cdot 1 - \frac{1}{x+3} \cdot 1$

$= \frac{1}{x+2} + \frac{1}{x-2} - \frac{1}{x+3}$

example: $y = \log_2 x$

$\therefore 2^y = x$

$\ln(2^y) = \ln x$

$y\ln 2 = \ln x$

$y = \frac{1}{\ln 2} \cdot \ln x$

$$y = \underbrace{\frac{1}{\ln 2}}_{\uparrow \text{ constant}} \cdot \frac{1}{x}$$

INTEGRATION WITH THE LOG FUNCTION

if $y = \ln x$

$y' = \frac{1}{x}$

then $\int \frac{1}{x}\,dx$

$= \ln x + c$

x^{-1} — cannot work as $n \neq -1$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1}$$

if $y = \ln[f(x)]$

$y' = \frac{f'(x)}{f(x)}$

$\therefore \int \frac{f'(x)}{f(x)}$

$= \ln[f(x)] + c$

example: $\int \frac{4}{x-1}\,dx$

↳ find dy/dx of this $= 1$

$4 \int \frac{1}{x}$

↑ take 4 out

$\therefore 4\ln(x-1) + c$

example: $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

↖ f' of this $= 2x$

$= \frac{1}{2}\ln(x^2+1) + c$

example: $\int \frac{2x-3}{x^2-3x+4}$

↖ f' of this $= 2x-3$

$= \ln(x^2-3x+4) + c$

example: $\frac{1}{2}\int \frac{2(3x-2)}{3x^2-4x+5}\,dx$

↳ $f' = 6x-4$

$= \frac{1}{2}\ln(3x^2-4x+5) + c$

example: $\int \frac{x}{x+1}\,dx$

↳ $f' = 1$ cannot work yet

$$\begin{array}{r|l} & 1 \\ \hline x+1 & x \\ & -x+1 \\ \hline & -1 \end{array}$$

ignore this

↑ remainder

$\frac{x}{x+1} = 1 +$ remainder

$\frac{x}{x+1} = 1 - \frac{1}{x+1}$

now $\int 1 - \frac{1}{x+1}\,dx$

$= x - \ln(x+1) + c$

example: $\int \frac{\sin x}{1-\cos x}\,dx$

↳ $f' = \sin x$

$= \ln(1-\cos x) + c$

DEFINITE INTEGRALS

example: $\frac{1}{5}\int_1^4 \frac{1 \times 5}{5x-2}\,dx$

↳ $f' = 5$

$= \frac{1}{5}\Big[\ln(5x-2)\Big]_1^4$

$= \frac{1}{5}[\ln 18 - \ln 3]$

$= \frac{1}{5}\ln\frac{18}{3}$

$= \frac{1}{5}\ln 6$

example: $\int_1^2 \frac{3x^2-4x}{x^3-2x^2+5}$

↳ $f' = 3x^2-4x$

$= \Big[\ln(x^3-2x^2+5)\Big]_1^2$

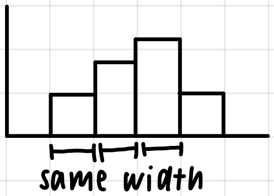
$= [\ln 5 - \ln 4]$

$= \ln\left(\frac{5}{4}\right)$

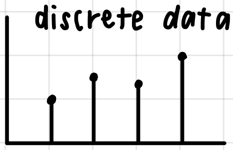
CRV's

histogram

- column graph without spaces



- measured e.g. height, time, weight etc.



- counted

consider 17 different weights (to nearest kg)

74, 88, 75, 82, 60, 75, 85, 65, 90, 69, 75, 70, 71, 65, 88, 72, 67

GRAPHED FREQUENCY TABLE

mass (M)	f	relative f	midpoints
$60 \leq x < 65$	1	$1/17$	62.5
$65 \leq x < 70$	4	$4/17$	67.5
$70 \leq x < 75$	4	$4/17$	72.5
$75 \leq x < 80$	3	$3/17$	77.5
$80 \leq x < 85$	1	$1/17$	82.5
$85 \leq x < 90$	3	$3/17$	87.5
$90 \leq x < 95$	1	$1/17$	92.5
	total: 17	$\Sigma = 1$	

x estimate

might only be given these 2 columns

a) $P(M < 80)$

$3/17 + 4/17 + 4/17 + 1/17$

$= 12/17$

b) $P(75 \leq M < 90)$

$= 3/17 + 1/17 + 3/17$

$= 7/17$

c) $P(75 \leq M < 90 \mid M < 80) = \dfrac{P(75 \leq M < 80)}{P(M < 80)}$

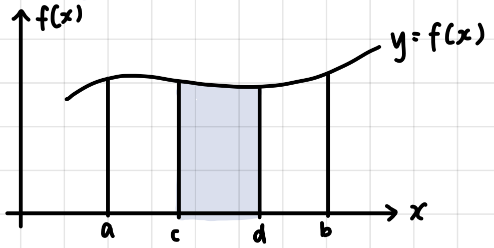
$$\frac{3/17}{3/17 + 4/17 + 4/17 + 1/17}$$

$$= \frac{3/17}{12/17} = \frac{3}{12} = \frac{1}{4}$$

d) $P(M = 80) = 0$

↑ must be an interval

PROBABILITY DENSITY FUNCTION (PDF)



$f(x) = \quad a \leq x \leq b$

$P(c \leq x \leq d) = \int_c^d f(x)\,dx$

probability = area under the curve

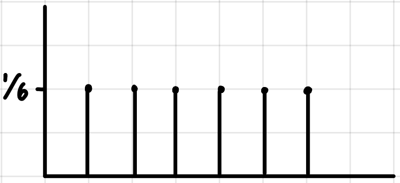
also $\int_a^b f(x)\,dx = 1 \rightarrow$ sum of prob = 1

$f(x) \geq 0$ for $a \leq x \leq b$

now $P(x = e) = \int_e^e f(x)\,dx = 0$

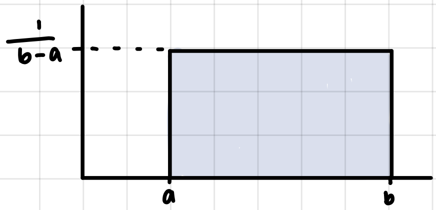
	DRV	CRV
M (mean)	$\Sigma x \cdot P(x)$	$\int_{-\infty}^{\infty} x \cdot P(x)\,dx$
δ^2 (variance)	$\Sigma (x - M)^2 \cdot P(x)$	$\int_{-\infty}^{\infty} (x - M)^2 \cdot P(x)\,dx$

UNIFORM DISTRIBUTION



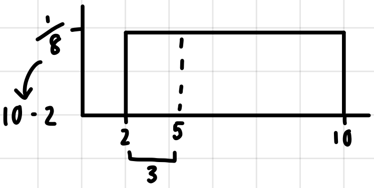
"rolling a dice"

RECTANGULAR DISTRIBUTION



pdf: $P(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

example: a uniform CRV over the interval $2 \le x \le 10$



a) $P(x \le 5)$

a.k.a 2 → 5

$\frac{1}{8} \times 3 = 3/8$

b) $P(3 \le x \le 8)$

$\frac{1}{8} \times 5 = 5/8$

c) $P(x = 5)$

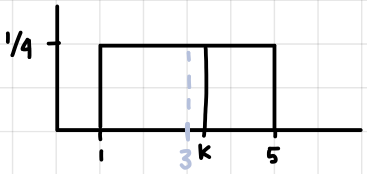
$= 0$

d) $P(x = 5 \mid 3 \le x \le 8)$

$= \frac{P(3 \le x \le 5)}{5/8}$

$= \frac{\frac{1}{8} \times 2}{5/8} = 2/5$

example: CRV below, determine k if $P(x \geq k \mid x \geq 3) = 0{\cdot}5$



k lies between 3 → 4

$\frac{P(k \le x \le 5)}{P(x \geq 3)} = \frac{1}{2}$

$\frac{\frac{1}{\cancel{4}}(5-k)}{\cancel{1/4} \times \cancel{2}} = \frac{1}{\cancel{2}}$

$5 - k = 1$

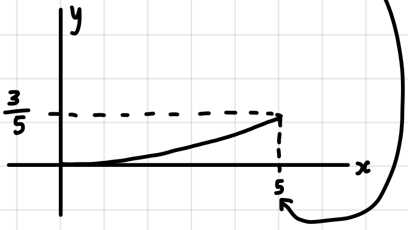
$\therefore k = 4$

NON UNIFORM DISTRIBUTIONS

example: X in a CRV w/ pdf

$f(x) = \frac{3x^2}{125} \quad 0 \le x \le 5$

a) sketch f



b) verify it is a pdf

$f(x) \geq 0$ ✓

$\frac{3}{125}\int_0^5 x^2 = \frac{\cancel{3}}{125} \cdot \frac{1}{\cancel{3}} [x^3]_0^5$

area under the curve = 1

$= \frac{5^3}{125} = 1$ ✓

c) $P(1 \le x \le 3)$

$\frac{3}{125}\int_1^3 x^2 \, dx$

$= \frac{\cancel{3}}{125} \cdot \frac{1}{\cancel{3}} [x^3]_1^3$

$= \frac{1}{125} [27] - [1]$

$= \frac{26}{125}$

d) $P(x \le 3)$

$= \frac{3}{125}\int_0^3 x^2 dx$

$= \frac{\cancel{3}}{125} \cdot \frac{1}{\cancel{3}} [x^3]_0^3$

$= \frac{1}{125}[27] = \frac{27}{125}$

e) $P(x \geq 1 \mid x \le 3)$

$\frac{26/125}{27/125} = \frac{26}{27}$

f) determine $\overset{\nearrow E(X)}{\mu}$ & σ^2

$\mu = E(X) = \frac{3}{125}\int_0^5 x^3 dx$

$= \frac{3}{125} \cdot \frac{1}{4} [x^4]_0^5$

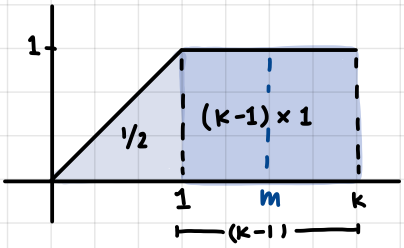
$= \frac{3 \times 5^{\cancel{4}\,1}}{\cancel{125} \times 4}$

$= \frac{3 \times 5}{4} = \frac{15}{4}$

$\sigma^2 = \int_0^5 (x - \frac{15}{4})^2 \times (\frac{3x^2}{125}) dx$

use calculator: 15/16

example: pdf $f(x)$ is defined for $0 \leq x \leq k$ and is sketched below



a) determine k

$\frac{1}{2} + (k-1) = 1$

$k - 1 = \frac{1}{2}$

$k = \frac{3}{2}$

b) define pdf

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \leq \frac{3}{2} \end{cases}$$

c) $P(x \leq 1)$ ↳ area

$= \frac{1}{2}$

d) determine m such that $P(x \leq m) = 3/4$

$= \frac{1}{2} + (m-1)(1)$

↑ area of m rectangle

$= \frac{1}{2} + (m-1) = \frac{3}{4}$

$= \frac{3}{4} - \frac{1}{2} = m - 1$

$= m = \frac{1}{4} + 1$

$= m = 5/4$

example: $f(x) = \begin{cases} kx^3, & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$

a) determine k

$k \int_0^4 x^3 \, dx = 1$

$\therefore \frac{k}{4} [x^4]_0^4 = 1$

$\frac{k}{\cancel{4}} \cdot 4^{\cancel{4}\,3} = 1$

$64k = 1$

$\therefore k = \frac{1}{64}$

b) $P(x \leq 3 \mid x \geq 1)$

$= \frac{P(1 \leq x \leq 3)}{P(x \geq 1)}$

$= \frac{\cancel{\frac{1}{64}} \int_1^3 x^3 dx}{\cancel{\frac{1}{64}} \int_1^4 x^3 dx}$

$= \frac{\frac{1}{4}[x^4]_1^3}{\frac{1}{4}[x^4]_1^4}$

$= \frac{81 - 1}{256 - 1}$

$= \frac{80}{255}$

c) $P(x \leq m) = \frac{3}{4}$, determine m

$\int_0^m \frac{x^3}{64} \, dx = \frac{3}{4}$

solve on classpad

example: the life (X) in yrs of an electronic component as pdf $f(x) = a - \frac{1}{2}(x-1)^2$, $0 \leq x \leq 2$

· pos· prob·
· area = 1

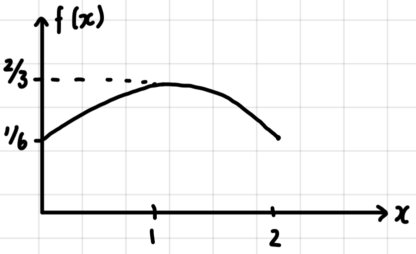
a) determine 'a'

solve: $\left[a - \frac{1}{2}(x-1)^2\right]_0^2 = 1$

$a = 2/3$

b) sketch f

$f(x) = \frac{2}{3} - \frac{1}{2}(x-1)^2$



c) determine the median

= middle ∴ 1 (symmetrical graph)

or $\int_0^m f(x) \, dx = \frac{1}{2}$ ← solve this

d) calculate the mean & s·d for x

$\sqrt{\text{var}}$ ↗

$E(X) = \mu = \int_0^2 x \left[\frac{2}{3} - \frac{1}{2}(x-1)^2\right] dx$

solve on classpad $= 1$

$\delta^2 = \int_0^2 (x-1)^2 \left[\frac{2}{3} - \frac{1}{2}(x-1)^2\right] dx$

$= 11/45$

$s \cdot d \cdot = \sqrt{11/45}$

LINEAR TRANSFORMATIONS (CRV)

consider a random continuous variable X with pdf $(f(x))$ for $a \leq x \leq b$
if the random variable $T = mX + n$
$m + n$ = constants

$$\left.\begin{array}{l} \text{then } E(T) = mE(X) + n \\ \text{and } var(T) = m^2 var(X) \end{array}\right\} \text{same as DRV's}$$

example:
if a random variable X has p.d.f

$$f(x) = \frac{\sqrt{x}}{18} \ , \quad 0 \leq x \leq 9$$

determine μ + δ (s·d·) for X, hence calculate the mean + s·d· for T if $T = 2x + 5$

$$\mu = \overset{\swarrow \text{exact value}}{\frac{27}{5}} \ (5\ 4)$$

$$\int_0^9 \frac{\sqrt{x}}{18} \cdot x$$

$$\delta = \frac{18\sqrt{21}}{35} \quad (2·357 \ \ 3d·p)$$

$T = 2x + 5$

$E(T) = 15·8$

$2 \times 5·4 + 5$

$s·d(T) = 4·714$

$2 \times 2·537$

CUMULATIVE FUNCTION

on calc

PD : $x =$

CD · $x \le$

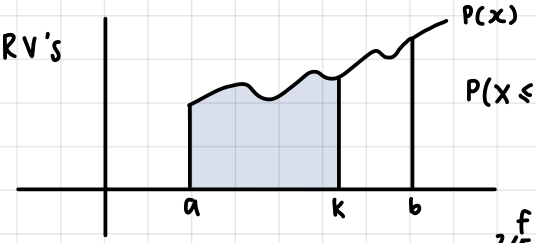
DRV's

x	0	1	2
$P(X=x)$	0·1	0·3	0·6

recall

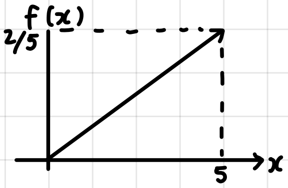
x	0	1	2
$P(X\le x)$	0·1	0·4	1

CRV's



$$P(X \le k) = \int_a^k p(x)\,dx$$

example: $P(x) = \begin{cases} \frac{2x}{25} & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$



$\frac{2}{5}$ (when $x = 5$

a) determine the CDF (F(x)) ← $P(X \le x)$

$$F(x) = \frac{2}{25}\int_0^x x\,dx$$

$$= \frac{2}{25} \cdot \frac{x^2}{2}$$

$$= \frac{2}{25} \cdot \frac{1}{2}\left[x^2\right]_0^x$$

$$= \frac{1}{25}x^2$$

piece-wise

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{25}x^2 & 0 \le x \le 5 \\ 1 & x > 5 \end{cases}$$

↖ since it is cumulative

b) $P(X \le 3)$

sub $x = 3$ into $\frac{1}{25}x^2$

$f(3) = \frac{1}{25}(3^2)$

$= 9/25$

c) $P(2 \le x \le 3)$

$= F(3) - F(2)$

$= 9/25 - 4/25$

$= 5/25$

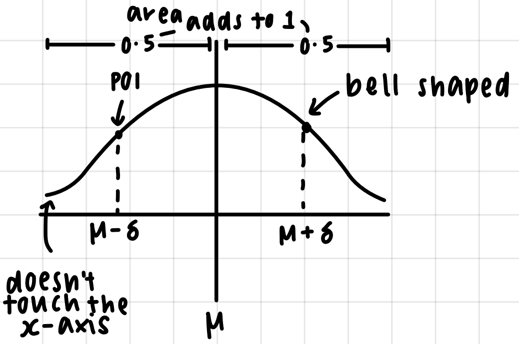
$= 1/5$

NORMAL DISTRIBUTION

pdf:

$$f(x) = \underbrace{\frac{1}{\sigma\sqrt{2\pi}}}_{\text{scaling factor (makes area = 1)}} e^{-\frac{1}{2}\underbrace{\left(\frac{x-\mu}{\sigma}\right)}_{\text{z score}}^{2}} \text{ — makes it symmetrical}, \quad -\infty < x < \infty$$

cannot be $\infty \leq$ as cannot $= \infty$

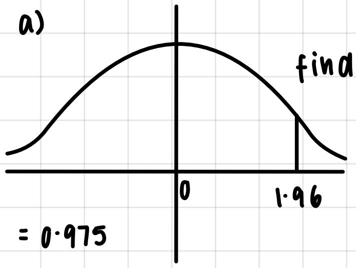


If x is a normal variable with mean μ and s.d. σ then $X \sim N(\mu, \sigma^2)$ ← variance

if $\mu = 0$ and $\sigma = 1$, then X is a standard normal variable i.e. $Z \sim N(0,1)$

example: if $Z \sim N(0,1)$ determine

a)



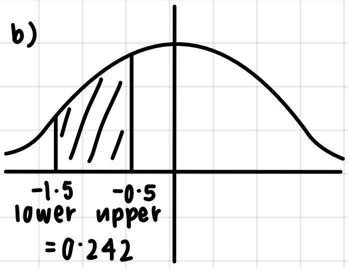
$= 0.975$

finding area to the left:

on calc: NormCD

lower $-\infty$
upper 1.96
$\sigma = 1$
$\mu = 1$

b)



$= 0.242$

STANDARD (Z) SCORES)

$$z = \frac{x - \mu}{\sigma}$$

x ← raw score

example:

		μ	σ	
test 1	65.38%	63.37	12.74	$z_1 = 0.1578$
test 2	68.42%	66 19	13.13	$z_2 = 0.1698$

how many s.d. above the mean

bigger = higher z score = further from μ

example: if X is normal, $X \sim N(100, 25)$

calculate $P(X \leq 101)$

calc:
upper 101
lower $-\infty$
$\sigma = 5$ ← since $5^2 = 25$
$\mu = 100$

$= 0.5793$

alt. method

$P = \left(z < \frac{101-100}{5}\right)$

$P = (z \leq 0.2)$

calc:
s.d. = 1
$\mu = 0$
upper = 0.2
= 0.5793

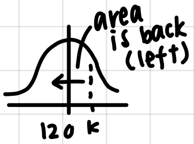
example: if $X \sim N(120, 20^2)$

a) determine k if $P(X \leq k) = 0.75$

on calc: inverse distribution

tail setting: left

$k = 133.5$



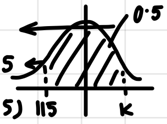
b) $P(120 - k < X < 120 + k) = 0.9$

prob = 0.9
$\sigma = 20$
$\mu = 120$

$120 - k = 87.102$ or $120 + k =$ upper

$k = 32.9$

c) $P(115 \leq X \leq k) = 0.5$



$P(X \leq k) - P(X \leq 115) = 0.5$

$P(X \leq k) = 0.5 + P(X \leq 115)$

lower $-\infty$
upper 115
σ 20
$\mu = 120$

$= 0.40129$

$P(X \leq k) = 0.90129$

inverse:

prob: .90129
$\sigma = 20$
$\mu = 120$
$\therefore k = 145.8$

example: If $X \sim N(\mu, 16)$, determine μ if $P(X \geq 20) = 0.01$

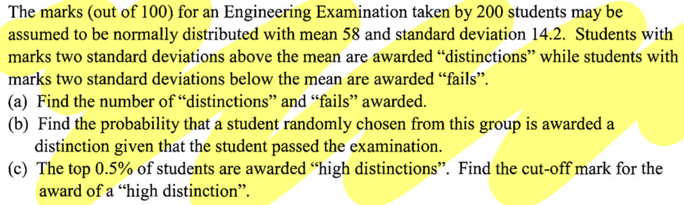
$z = \frac{x - \mu}{\sigma}$

i.e. $P\left(z \geq \underbrace{\frac{20-\mu}{4}}_{z}\right) = 0.01$

inverse, right ($\geq$)

$\therefore \frac{20-\mu}{4} = 2.3263$

$\mu = 10.69$



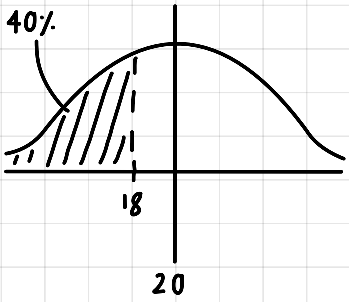
The marks (out of 100) for an Engineering Examination taken by 200 students may be assumed to be normally distributed with mean 58 and standard deviation 14.2. Students with marks two standard deviations above the mean are awarded "distinctions" while students with marks two standard deviations below the mean are awarded "fails".

(a) Find the number of "distinctions" and "fails" awarded.

(b) Find the probability that a student randomly chosen from this group is awarded a distinction given that the student passed the examination.

(c) The top 0.5% of students are awarded "high distinctions". Find the cut-off mark for the award of a "high distinction".

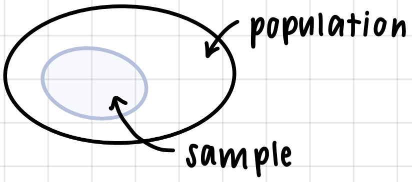
50% = pass



18 is the quantile such that 40% of the distribution is below this value
i.e. we say 18 is the 0·4 quantile

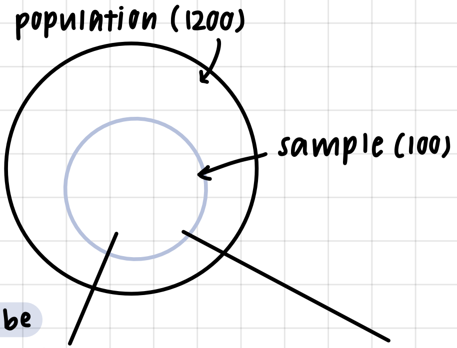
SAMPLING + PROPORTIONS

population - all measurements being studied
sample - subset of the population



TYPES OF SAMPLING

consider a population of students - survey of the ways students get to school



- could survey all students (too large)

can be

random or **not random**

random
- simple random sampling
  (e.g. assign a no. to students & random number generator)
  (could be biased)
- stratified sample
  (e.g. put into groups Yr 7, Yr 8 etc.)
  (could also be biased)
- cluster groups
  (e.g. students in class)

not random

VOLUNTARY SAMPLE
- go on website
  (bias)

CONVENIENCE SAMPLE
- survey first 100 students who arrive at school (bias)

BIAS

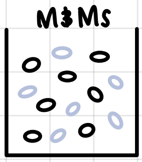
to remove bias, the sample needs to be as random as possible

UNDER COVERAGE
- occurs when members of the population are inadequately represented
e.g. London (1936) pulled names from the phone book - bias - since poor people had no phone

NON-RESPONSE BIAS
- when people chosen are unwilling or unable to participate
e.g. mail surveys - sent out but not necessarily returned

OTHERS

PROPORTIONS



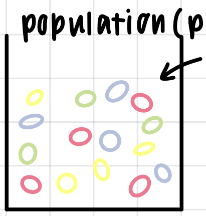
proportion of blue M&Ms
$p = 0.6$
↑ population parameter

1 trial

define Bernoulli RV

$$y = \begin{cases} 1 & , \text{blue} \\ 0 & , \text{not blue} \end{cases}$$

$\mu_y = p = 0.6$
$\delta_y = \sqrt{p(1-p)}$
$= \sqrt{0.6(1-0.6)}$



sample proportion = 10 ($\hat{p}$)

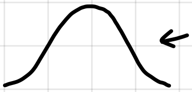
$\hat{p} = \frac{x}{n}$ ← successful

$= \frac{x}{10}$

e.g. $\frac{x}{10} = 0.3 \ldots$
$\frac{x}{10} = 0.7 \ldots$] do a bunch of trials

now $x = \Sigma$ 10 independent trials of y
↑ now binomial

$\mu_x = np \quad = 10 \times 0.6$
$\delta_x = \sqrt{np(1-p)} = \sqrt{10 \times 0.6(1-0.6)}$



← will tend towards a normal for more samples

DISTRIBUTION OF THE SAMPLE PROPORTION

$\mu_{\hat{p}} = \frac{\mu_x}{n} = \frac{np}{n} = p$

mean of same proportion = proportion itself

$\delta_{\hat{p}} = \frac{\delta_x}{n} = \frac{\sqrt{n(p)(1-p)}}{n} = \sqrt{\frac{np(1-p)}{n^2}} = \sqrt{\frac{p(1-p)}{p}}$

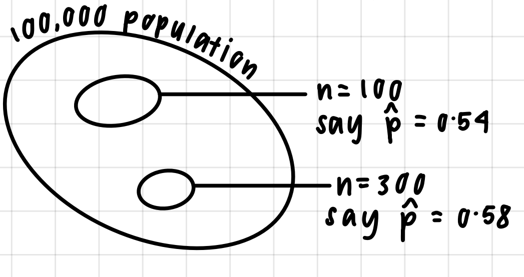
how good is the normal fit for a distribution of a sample proportion

rule of thumb:
$n \geqslant 30$
if $np \geqslant 10$ & $n(1-p) \geqslant 10$
= reasonably good fit

reduce error by ↑ sample size

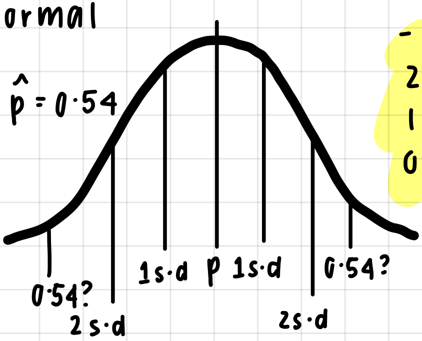
situation: election

candidate A vs candidate B
what is the likelihood of A winning the election?



sampling - distribution of the sample proportions for 100 ≈ normal

case 1: $\hat{p} = 0.54$



-2 2 high
2 2 low
1
0

what is the probability that $\hat{p} = 0.54$ is within $2\delta_{\hat{p}}$ of p?
≈ 95% of the area is within 2δ of the mean
there is a 95% probability that the population proportion (p) is within $\underline{2\delta_{\hat{p}}}$ of $\hat{p}$

if we know that this value is, then we would be able to calculate a confidence integral
<u>we don't know p</u> but we can use $\hat{p}$

new statistic:
standard error $\hat{p} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
↑
unbiased estimator ($\delta_{\hat{p}}$)

$= \sqrt{\frac{0.54(1-0.54)}{180}} \approx 0.05$

cont. next pg ⟶

95% confidence intergral

0·54 − 2 × 0·05 and 0·54 + 2 × 0·05

0·44 0·64

every time new $\hat{p}$

= new confidence integral

sometimes p will fall into it, sometimes it won't